

4.2 Building Trigonometry from an Integral

WORKSHEET

The question

In this question, you are not permitted to use any properties of trigonometric functions or inverse trigonometric functions.

The function T is defined for $x > 0$ by

$$T(x) = \int_0^x \frac{1}{1+u^2} du,$$

and $T_\infty = \int_0^\infty \frac{1}{1+u^2} du$ (which has a finite value).

(i) By making an appropriate substitution in the integral for $T(x)$, show that

$$T(x) = T_\infty - T(x^{-1}).$$

(ii) Let $v = \frac{u+a}{1-au}$, where a is a constant. Verify that, for $u \neq a^{-1}$,

$$\frac{dv}{du} = \frac{1+v^2}{1+u^2}.$$

Hence show that, for $a > 0$ and $x < \frac{1}{a}$,

$$T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a).$$

Deduce that

$$T(x^{-1}) = 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$$

and hence that, for $b > 0$ and $y > \frac{1}{b}$,

$$T(y) = 2T_\infty - T\left(\frac{y+b}{by-1}\right) - T(b).$$

(iii) Use the above results to show that $T(\sqrt{3}) = \frac{2}{3}T_\infty$ and $T(\sqrt{2}-1) = \frac{1}{4}T_\infty$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ? **Ask yourself** Which trigonometric identities do parts (i) and (ii) correspond to? (Think of $\arctan x + \arctan \frac{1}{x}$, and the formula for $\tan(A + B)$.)

STAGE 2 The addition substitution

- ? **Ask yourself** Differentiate $v = \frac{u + a}{1 - au}$ by the quotient rule. Then compute $1 + v^2$ as a single fraction and compare.

- ? **Ask yourself** This says $\frac{du}{1 + u^2} = \frac{dv}{1 + v^2}$. So the substitution $u \mapsto v$ doesn't change the integrand at all. What does it do to the limits of $T(x)$?

STAGE 4 Special values

- 🕒 **Pause and try** Choose y and b in the last identity so that the argument $\frac{y+b}{by-1}$ equals y itself. What do you get?

- ? **Ask yourself** For $\sqrt{2} - 1$, try the addition identity with $x = a = \sqrt{2} - 1$. What is $\frac{2x}{1-x^2}$? And what is $T(1)$?

Try this next

- The exponential and the cosine.** (From the STEP II Specimen paper.) *Without using standard properties of exponential or trigonometric functions:* a function f satisfies $f' = f$ with $f(0) = 1$. Show that $\frac{d}{dx} [f(a-x)f(x)] = 0$ and deduce that $f(x)f(y) = f(x+y)$. Functions c and s satisfy $c' = -s$, $s' = c$, $s(0) = 0$, $c(0) = 1$. Show that $c(x+y) = c(x)c(y) - s(x)s(y)$. (Hint: fix y and consider $c(x+y) - c(x)c(y) + s(x)s(y)$ and a similar combination for s ; show the sum of their squares is constant.)
- The inverse sine.** (From STEP III, 2007.) Let $S(x) = \int_0^x \frac{du}{\sqrt{1-u^2}}$. Using $y = \frac{u}{\sqrt{1+u^2}}$, show that $T(x) = S\left(\frac{x}{\sqrt{1+x^2}}\right)$ and deduce that $S\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}T_\infty$.
- An early Machin-type formula.** Use the addition identity to show that $T\left(\frac{1}{2}\right) + T\left(\frac{1}{3}\right) = T(1) = \frac{1}{2}T_\infty$.

- 4 T_∞ is finite. Show that $T(x) \leq 1 + \int_1^x \frac{du}{u^2} < 2$ for all $x > 0$, and deduce from the monotone convergence theorem (Section 2.2) that T_∞ exists.

Further reading

- **Tom Körner**, *Calculus for the Ambitious* (Cambridge University Press). Builds the elementary functions rigorously, discussing the different approaches.
- **Walter Rudin**, *Principles of Mathematical Analysis* (McGraw-Hill). The famous “baby Rudin”; the prologue to Chapter 8 constructs $\exp$, $\sin$ and $\cos$ from power series in a few elegant pages. Advanced.
- **Victor Prasolov & Yuri Solov'yev**, *Elliptic Functions and Elliptic Integrals* (American Mathematical Society). For where the lemniscate leads. Advanced.

The full guided solution, and what the question leads to at university, is in Section 4.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/trig-from-integral.html