

4.1 What Is an Integral?

WORKSHEET

The question

(i) Prove that

$$\sum_{r=1}^n r(r+1)(r+2)(r+3)(r+4) = \frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5)$$

and deduce that

$$\sum_{r=1}^n r^5 < \frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

(ii) Prove that, if $n > 1$,

$$\sum_{r=0}^{n-1} r^5 > \frac{1}{6}(n-5)(n-4)(n-3)(n-2)(n-1)n.$$

(iii) Let f be an increasing function. If the limits

$$\lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{a}{n} f\left(\frac{ra}{n}\right) \quad \text{and} \quad \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{a}{n} f\left(\frac{ra}{n}\right)$$

both exist and are equal, the definite integral $\int_0^a f(x) dx$ is defined to be their common value. Using this definition, prove that

$$\int_0^a x^5 dx = \frac{1}{6}a^6.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself The two sums in part (iii) are sums of areas of rectangles of width $\frac{a}{n}$. For an increasing function f , which sum is an underestimate of the area under the curve, and which an overestimate?

STAGE 1 A sum that telescopes

Ask yourself Write $P(r) = r(r+1)(r+2)(r+3)(r+4)(r+5)$, a product of six consecutive integers. What is $P(r) - P(r-1)$?

STAGE 2 A lower bound

Pause and try Adapt Stage 1 to a product of five consecutive integers going down: $r(r-1)(r-2)(r-3)(r-4)$. Find its sum from $r = 0$ to $n-1$, and compare it with r^5 .

STAGE 3 The integral

Ask yourself Write the lower and upper sums for $f(x) = x^5$ in terms of $\sum r^5$. What power of n appears in the denominator?

Try this next

1 A limit as an integral. (From STEP III, 1999.) Justify by a sketch that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n f\left(1 + \frac{m}{n}\right) = \int_1^2 f(x) \, dx.$$

Deduce that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \ln 2,$$

and evaluate $\lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2} \right)$.

2 Archimedes' parabola. Using $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$ and the definition in part (iii), prove that $\int_0^a x^2 \, dx = \frac{1}{3}a^3$.

3 Rising factorials. Show that $\sum_{r=1}^n r(r+1) \cdots (r+k-1) = \frac{n(n+1) \cdots (n+k)}{k+1}$ for

every positive integer k .

- 4 The exact sum.** Use the identity $r(r+1)(r+2)(r+3)(r+4) = r^5 + 10r^4 + 35r^3 + 50r^2 + 24r$ and the known formulas for $\sum r^k$ with $k \leq 4$ to find $\sum_{r=1}^n r^5$ exactly.

Further reading

- **David Bressoud, *A Radical Approach to Lebesgue's Theory of Integration*** (Cambridge University Press). The story from Riemann's integral to Lebesgue's, told through the problems that forced each step.
- **Michael Spivak, *Calculus*** (Cambridge University Press). Chapter 13 defines the integral by upper and lower sums, exactly as here.
- **Graham, Knuth & Patashnik, *Concrete Mathematics*** (Addison-Wesley). Chapter 2 develops the calculus of finite differences, with rising and falling factorials.

The full guided solution, and what the question leads to at university, is in Section 4.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/riemann.html