

3.9 Inventing a New Function

WORKSHEET

The question

- (i) Sketch the curve $y = xe^x$, giving the coordinates of any stationary points.
- (ii) The function f is defined by $f(x) = xe^x$ for $x \geq a$, where a is the minimum possible value such that f has an inverse function. What is the value of a ?
Let g be the inverse of f . Sketch the curve $y = g(x)$.
- (iii) For each of the following equations, find a real root in terms of a value of the function g , or demonstrate that the equation has no real root. If the equation has two real roots, determine whether the root you have found is greater than or less than the other root.
- (a) $e^{-x} = 5x$ (b) $2x \ln x + 1 = 0$ (c) $3x \ln x + 1 = 0$ (d) $x = 3 \ln x$
- (iv) Given that the equation $x^x = 10$ has a unique positive root, find this root in terms of a value of the function g .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** What does it actually mean to “solve” an equation? Is $x = \sqrt{2}$ a solution of $x^2 = 2$, or just a name for the solution?

STAGE 1 The curve $y = xe^x$

? **Ask yourself** What happens to xe^x as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$? Where does the curve cross the axes?

STAGE 3 Four equations

? **Ask yourself** For (b) and (c), the unknown appears as $x \ln x$. What substitution turns $x \ln x$ into something of the form ue^u ?

/ **Pause and try** Now try (d), $x = 3 \ln x$. (*Hint:* exponentiate and aim for (something) $e^{\text{something}}$ with the something being a multiple of x .)

Try this next

1 Two, four, and one more. The equation $x^2 = 2^x$ has the obvious solutions $x = 2$ and $x = 4$ (compare Section 3.5). Show that it also has a negative solution, and that this solution is

$$x = -\frac{2}{\ln 2} \mathfrak{g}\left(\frac{\ln 2}{2}\right) \approx -0.767.$$

2 A derivative. By differentiating $g(x)e^{g(x)} = x$ implicitly, show that, for $x \neq 0$,

$$g'(x) = \frac{g(x)}{x(1 + g(x))}.$$

What happens at $x = -\frac{1}{e}$?

3 An integral. Use the substitution $x = we^w$ to show that $\int_0^e g(x) \, dx = e - 1$.

4 The tower. Assuming the tower $T = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}$ converges, show that $T = \sqrt{2}^T$, and deduce that $T = 2$ or $T = 4$. Which is it? (Compute a few terms of the sequence $t_1 = \sqrt{2}$, $t_{n+1} = \sqrt{2}^{t_n}$, and think about the cobweb diagrams of Section 2.2.)

Further reading

- ▶ **Robert Corless et al.**, “**On the Lambert W function**”, *Advances in Computational Mathematics* 5 (1996). The paper that named the function. The first sections are readable with this section behind you.
- ▶ **Brian Hayes**, “**Why W?**”, *American Scientist* 93 (2005). A lively essay on the function’s history and uses. Free online.
- ▶ **Brian Conrad**, “**Impossibility theorems for elementary integration**”. Lecture notes, available online, explaining Liouville’s theory. Advanced.

The full guided solution, and what the question leads to at university, is in Section 3.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/lambert-w.html