

3.8 Functions from Their Addition Laws

WORKSHEET

The question

The definition of the derivative f' of a (differentiable) function f is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}. \quad (*)$$

- (i) The function f has derivative f' and satisfies

$$f(x+y) = f(x)f(y)$$

for all x and y , and $f'(0) = k$ where $k \neq 0$. Show that $f(0) = 1$.

Using $(*)$, show that $f'(x) = kf(x)$ and find $f(x)$ in terms of x and k .

- (ii) The function g has derivative g' and satisfies

$$g(x+y) = \frac{g(x) + g(y)}{1 + g(x)g(y)}$$

for all x and y , $|g(x)| < 1$ for all x , and $g'(0) = k$ where $k \neq 0$.

Find $g'(x)$ in terms of $g(x)$ and k , and hence find $g(x)$ in terms of x and k .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself The equation holds for *all* x and y . What is the most useful thing to try first?

STAGE 1 The value at zero



Ask yourself What goes wrong if $f(0) = 0$?

STAGE 2 The derivative everywhere from the derivative at zero



Ask yourself In $(*)$, write $f(x+h) = f(x)f(h)$. What can you take outside the limit?

STAGE 4 The second rule



Ask yourself Put $y = 0$ in the rule for g . What does that tell you about $g(0)$? Where does $|g(x)| < 1$ come in?



Pause and try Now compute $g(x+h) - g(x)$ using the rule, simplify, divide by h and let $h \rightarrow 0$.

STAGE 5 Solving it



Ask yourself Separate the variables. Which function has derivative $\frac{1}{1-g^2}$ for $|g| < 1$?

Try this next

- 1 A tangent rule.** Suppose g is differentiable, $g'(0) = k \neq 0$, and $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ whenever both sides are defined. Show that $g'(x) = k(1+g(x)^2)$ and deduce that $g(x) = \tan kx$ near $x = 0$.
- 2 Cauchy's equation.** If f is differentiable and $f(x+y) = f(x) + f(y)$ for all x, y , show that $f(x) = kx$.
- 3 Without derivatives.** **HARDER** Suppose only that $f(x+y) = f(x)f(y)$ and $f(1) = a > 0$.

Show that $f(n) = a^n$ for positive integers n , then that $f(1/n) = a^{1/n}$, and then that $f(q) = a^q$ for every positive rational q . (Why is continuity needed to go further?)

- 4 Rocket science.** A rocket can increase its speed by $0.5c$ relative to its current frame each time it fires. Starting from rest, what is its speed (relative to the start) after 2, 3 and 4 firings? Use rapidities.

Further reading

- **Christopher Small, *Functional Equations and How to Solve Them*** (Springer). An accessible introduction aimed at problem-solvers.
- **Edwin Taylor & John Wheeler, *Spacetime Physics*** (W. H. Freeman). The classic introduction to special relativity, using rapidity throughout.
- **Tom Körner, *Calculus for the Ambitious*** (Cambridge University Press). Defines the exponential and trigonometric functions from their properties, exactly in the spirit of this section.

The full guided solution, and what the question leads to at university, is in Section 3.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/addition-laws.html