

3.7 $\pi^2 \leq 10$

WORKSHEET

The question

This question concerns the inequality

$$\int_0^\pi (f(x))^2 dx \leq \int_0^\pi (f'(x))^2 dx. \quad (*)$$

- (i) Show that $(*)$ is satisfied in the case $f(x) = \sin nx$, where n is a positive integer.

Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $f(0) \neq 0$ or $f(\pi) \neq 0$.

- (ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $f(0) = f(\pi) = 0$.

By setting $f(x) = ax^2 + bx + c$, where a , b and c are suitably chosen, show that $\pi^2 \leq 10$.

By setting $f(x) = p \sin \frac{1}{2}x + q \cos \frac{1}{2}x + r$, where p , q and r are suitably chosen, obtain another inequality for π .

Which of these inequalities leads to a better estimate for π^2 ?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Why might a function that is zero at both ends of $[0, \pi]$ have to be steep if it is large in the middle? Why does the condition $f(0) = f(\pi) = 0$ matter?

STAGE 1 Sine waves

Ask yourself What is $\int_0^\pi \sin^2 nx \, dx$? And $\int_0^\pi \cos^2 nx \, dx$? (Use $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$.)

STAGE 2 The end conditions matter

Ask yourself Find a simple function with $f(0) \neq 0$ for which $(*)$ fails. Then find one with $f(0) = 0$ but $f(\pi) \neq 0$.

STAGE 3 A parabola

Ask yourself Which quadratic vanishes at $x = 0$ and $x = \pi$? (Up to a constant multiple, there is only one. Why doesn't the multiple matter?)

STAGE 4 A trigonometric choice

Ask yourself For $f(x) = p \sin \frac{1}{2}x + q \cos \frac{1}{2}x + r$, what conditions do $f(0) = 0$ and $f(\pi) = 0$ impose on p, q, r ?

Pause and try Compute $\int_0^\pi f^2$ and $\int_0^\pi (f')^2$ for this function. What inequality for π results?

Try this next

1 A quartic. Use $f(x) = x^2(\pi - x)^2$ in $(*)$. Show that it gives only $\pi^2 \leq 12$. Why is this test function worse than the parabola? (Compare its shape with $\sin x$.)

2 Two sine waves. Check $(*)$ directly for $f(x) = \sin x + \sin 2x$, computing both sides.

3 A different interval. Show by a substitution that if $g(0) = g(L) = 0$ then $\int_0^L g^2 \leq$

$\frac{L^2}{\pi^2} \int_0^L (g')^2$. What does this say about strings of different lengths?

- 4 **Rayleigh's method.** **HARDER** Take $f(x) = x(\pi - x) + \lambda x^2(\pi - x)^2$ and choose λ to make the bound on π^2 as good as possible. (The algebra is heavy; a computer algebra system helps.)

Further reading

- **T. W. Körner, *Fourier Analysis*** (Cambridge University Press). A wonderful, discursive book on Fourier series and their applications, written by a Cambridge analyst.
- **Hardy, Littlewood & Pólya, *Inequalities*** (Cambridge University Press). Chapter 7 discusses Wirtinger's inequality and its relatives.
- **Richard Feynman, *The Feynman Lectures on Physics***, Volume II, Chapter 19 ("The principle of least action"). Feynman's lecture on minimising integrals, with physical intuition throughout. Free online.

The full guided solution, and what the question leads to at university, is in Section 3.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/wirtinger.html