

3.6 A Bound on Slopes

WORKSHEET

The question

The function f is defined by

$$f(x) = ax^2 + bx + c.$$

Show that

$$f'(x) = f(1) \left(x + \frac{1}{2}\right) + f(-1) \left(x - \frac{1}{2}\right) - 2f(0)x.$$

If a , b and c are real and such that $|f(x)| \leq 1$ for $|x| \leq 1$, show that $|f'(x)| \leq 4$ for $|x| \leq 1$.

Find particular values of a , b and c such that, for the corresponding function f of the above form, $|f(x)| \leq 1$ for all x with $|x| \leq 1$ and $|f'(x)| = 4$ for some x satisfying $|x| \leq 1$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** The formula for $f'(x)$ uses only three values of f : at -1 , 0 and 1 . Why should three values be enough to determine the derivative of a quadratic?

STAGE 2 The bound

- Ask yourself** Take absolute values in the formula and use the triangle inequality, together with $|f(1)|, |f(-1)|, |f(0)| \leq 1$. What bound on $|f'(x)|$ do you get, as a function of x ?

- Pause and try** Show that $|x + \frac{1}{2}| + |x - \frac{1}{2}| + 2|x| \leq 4$ for $|x| \leq 1$. (Split into cases according to the signs.)

STAGE 3 An extreme example

- Ask yourself** For the bound to be attained at $x = 1$, every inequality in Stage 2 must be an equality. What values must $f(1)$, $f(-1)$ and $f(0)$ take, and with what signs?

Try this next

- The general idea.** Show that if f is a quadratic then $f(x) = f(-1)\frac{x(x-1)}{2} + f(0)(1-x^2) + f(1)\frac{x(x+1)}{2}$, and differentiate to recover the formula in the question.
- Uniqueness.** HARDER Show that $\pm(2x^2 - 1)$ are the only quadratics with $|f(x)| \leq 1$ on $[-1, 1]$ and $|f'(x)| = 4$ somewhere in $[-1, 1]$.
- Chebyshev polynomials.** Use $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$ to check that $T_3(x) = 4x^3 - 3x$, and verify that $T'_3(1) = 9$. Show that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, and find T_4 .
- Slopes of T_n .** By differentiating $T_n(\cos \theta) = \cos n\theta$ with respect to θ , show that $T'_n(\cos \theta) = \frac{n \sin n\theta}{\sin \theta}$, and deduce that $T'_n(1) = n^2$ by letting $\theta \rightarrow 0$.

Further reading

- ▶ **Lloyd N. Trefethen**, *Approximation Theory and Approximation Practice* (SIAM). A beautiful modern book about Chebyshev polynomials and interpolation, readable by a strong school student, with computer experiments throughout.
- ▶ **Peter Borwein & Tamás Erdélyi**, *Polynomials and Polynomial Inequalities* (Springer). For the full story of Markov's inequality and its relatives. Advanced.
- ▶ **George M. Phillips**, *Interpolation and Approximation by Polynomials* (Springer). An undergraduate text covering Lagrange interpolation, Chebyshev polynomials and Markov-type inequalities.

The full guided solution, and what the question leads to at university, is in Section 3.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/markov-bound.html