

3.5 π^e or e^π ?

WORKSHEET

The question

- (i) Find the value of m for which the line $y = mx$ touches the curve $y = \ln x$.
If instead the line intersects the curve when $x = a$ and $x = b$, where $a < b$, show that $a^b = b^a$. Show by means of a sketch that $a < e < b$.
- (ii) The line $y = mx + c$, where $c > 0$, intersects the curve $y = \ln x$ when $x = p$ and $x = q$, where $p < q$. Show by means of a sketch, or otherwise, that $p^q > q^p$.
- (iii) Show by means of a sketch that the straight line through the points $(p, \ln p)$ and $(q, \ln q)$, where $e \leq p < q$, intersects the y -axis at a positive value of y . Which is greater, π^e or e^π ?
- (iv) Show, using a sketch or otherwise, that if $0 < p < q$ and $\frac{\ln q - \ln p}{q - p} = e^{-1}$, then $q^p > p^q$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Taking logarithms, $a^b = b^a$ is the same as $b \ln a = a \ln b$. Rearrange this so that a appears only on one side and b only on the other.

STAGE 1 The tangent through the origin



Ask yourself If $y = mx$ touches $y = \ln x$ at $x = x_0$, what two equations must hold?

STAGE 2 Lines that cut the y -axis above the origin



Ask yourself Now the line is $y = mx + c$ with $c > 0$. What are $\frac{\ln p}{p}$ and $\frac{\ln q}{q}$ in terms of m and c ?

STAGE 3 Chords to the right of e



Ask yourself The chord from $(p, \ln p)$ to $(q, \ln q)$, with $e \leq p < q$, has gradient less than the gradient of the curve at p . Why? And why does that make its y -intercept positive?



Pause and try Use this, and Stage 2, to decide which is greater, π^e or e^π .

STAGE 4 Chords parallel to the tangent



Ask yourself The chord is parallel to the tangent at $(e, 1)$. Since $\ln x$ is concave, the curve lies below that tangent. Does the chord cut the y -axis above or below the origin?

Try this next

- Whole numbers.** (From STEP I, 1988.) Use the graph of $h(x) = \frac{\ln x}{x}$ to find all pairs of distinct positive integers m and n with $n^m = m^n$.
- More comparisons.** (From STEP II, 2025.) Use the graph to show that $3^\pi > \pi^3$, and that $(\frac{9}{4})^{\sqrt{5}} > \sqrt{5}^{9/4}$. Given that $1 < x < 2$, decide which is larger, x^{x+2} or $(x+2)^x$. (Hint: $h(2) = h(4)$.)

3 Using $e^2 < 8$. (From STEP II, 2025.) Show that $9^{\sqrt{2}} > \sqrt{2}^9$ and $3^{2\sqrt{2}} > (2\sqrt{2})^3$ are equivalent. Given that $e^2 < 8$, decide which of $9^{\sqrt{2}}$ and $\sqrt{2}^9$ is larger.

4 A one-line proof. Use $e^x > 1 + x$ for $x \neq 0$, with $x = \frac{\pi}{e} - 1$, to prove $e^\pi > \pi^e$ directly.

Further reading

- **Eli Maor, *e: The Story of a Number*** (Princeton University Press). The history of e , including the equation $x^y = y^x$.
- **Paul Nahin, *An Imaginary Tale*** (Princeton University Press). Not only about $\sqrt{-1}$: full of curious calculations of exactly this kind.
- **Michael Spivak, *Calculus*** (Cambridge University Press). Chapter 11 covers the use of derivatives to understand the shape of graphs with complete rigour.

The full guided solution, and what the question leads to at university, is in Section 3.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/pi-e-or-e-pi.html