

3.4 Areas That Prove Inequalities

WORKSHEET

The question

The following result applies to any function f which is continuous, has positive gradient and satisfies $f(0) = 0$:

$$ab \leq \int_0^a f(x) \, dx + \int_0^b f^{-1}(y) \, dy, \quad (*)$$

where f^{-1} denotes the inverse function of f , and $a \geq 0$ and $b \geq 0$.

- (i) By considering the graph of $y = f(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

- (ii) By taking $f(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Verify that equality holds under the condition you stated above.

- (iii) Show that, for $0 \leq a \leq \frac{1}{2}\pi$ and $0 \leq b \leq 1$,

$$ab \leq b \arcsin b + \sqrt{1-b^2} - \cos a.$$

Deduce that, for $t \geq 1$,

$$\arcsin(t^{-1}) \geq t - \sqrt{t^2 - 1}.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** $\int_0^a f(x) dx$ is the area between the curve and the x -axis. What area does $\int_0^b f^{-1}(y) dy$ represent on the same picture?

STAGE 1 The picture

? **Ask yourself** Draw the rectangle with corners $(0, 0)$ and (a, b) . Shade the region under the curve from 0 to a in one colour and the region to the left of the curve from 0 to b in another. Do the two regions cover the rectangle? When is there no overlap and no excess?

STAGE 2 Young's inequality

? **Ask yourself** With $f(x) = x^{p-1}$, what is $f^{-1}(y)$? Compute both integrals, and use $\frac{1}{p} + \frac{1}{q} = 1$ to tidy the second.

STAGE 3 An inverse sine

? **Ask yourself** $\int_0^a \sin x dx$ is easy. How would you integrate $\arcsin y$?

✓ **Pause and try** Now deduce the inequality for $\arcsin(t^{-1})$. Which value of a should you choose? (Try the simplest possible one.)

Try this next

- 1 The 1987 version.** (From STEP I, 1987.) Using the same picture with $y = \sin x$, and choosing s so that equality holds in $(*)$, show that $\int_0^t \arcsin y dy = t \arcsin t - (1 - \cos(\arcsin t))$ for $0 \leq t \leq 1$. Check the result by differentiating both sides with respect to t .
- 2 Cauchy–Schwarz.** Use Young's inequality with $p = q = 2$ to prove $a_1 b_1 + a_2 b_2 + a_3 b_3 \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}$ for positive numbers. (*Hint:* first deal with the case where both square roots equal 1.)

3 An exponential version. Take $f(x) = e^x - 1$ in (*) to show that, for $a, b \geq 0$,

$$ab \leq e^a - 1 - a + (1 + b) \ln(1 + b) - b.$$

When is there equality?

4 How good is the bound? HARDER Use a calculator to compare $\arcsin(1/t)$ with $t - \sqrt{t^2 - 1}$ for $t = 1, 2, 10$. Show that for large t , $t - \sqrt{t^2 - 1} \approx \frac{1}{2t}$ while $\arcsin(1/t) \approx \frac{1}{t}$. Why is the bound off by a factor of 2? (Look at the picture with $a = 0$: how much of the excess is wasted?)

Further reading

- **J. Michael Steele, *The Cauchy–Schwarz Master Class*** (Cambridge University Press). Chapters 1 and 9 cover Cauchy–Schwarz and Hölder, with Young’s inequality along the way.
- **Roger Nelsen, *Proofs Without Words*** (Mathematical Association of America). Includes Young’s inequality and many other picture proofs.
- **V. I. Arnold, *Mathematical Methods of Classical Mechanics*** (Springer). Section 14 explains the Legendre transform with exactly this picture. Advanced, but the relevant section is short.

The full guided solution, and what the question leads to at university, is in Section 3.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/areas-inequalities.html