

3.3 The Power of Convexity

WORKSHEET

The question

A function $f(x)$ is said to be concave on some interval if $f''(x) < 0$ in that interval. Show that $\sin x$ is concave for $0 < x < \pi$ and that $\ln x$ is concave for $x > 0$.

Let $f(x)$ be concave on a given interval and let $x_1, x_2, \dots, x_n$ lie in the interval. *Jensen's inequality* states that

$$\frac{1}{n} \sum_{k=1}^n f(x_k) \leq f\left(\frac{1}{n} \sum_{k=1}^n x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = \dots = x_n$. You may use this result without proving it.

(i) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

(ii) By choosing a suitable function f , prove that

$$\sqrt[n]{t_1 t_2 \cdots t_n} \leq \frac{t_1 + t_2 + \cdots + t_n}{n}$$

for any positive integer n and for any positive numbers $t_1, t_2, \dots, t_n$.

Hence:

- (a) show that $x^4 + y^4 + z^4 + 16 \geq 8xyz$, where x, y and z are any positive numbers;
- (b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** Draw a curve that bends downwards, such as $y = \sin x$ on $(0, \pi)$.
 ? Mark two points on it and join them with a straight line. Is the chord above or below the curve?

STAGE 2 Triangles

- Ask yourself** Which function, which points, and what is their average? Do the angles of a triangle lie in an interval where the function is concave?

STAGE 3 AM–GM in one line

- Ask yourself** AM–GM compares a product with a sum. Which concave function turns products into sums?

STAGE 4 Using AM–GM

- Ask yourself** For (a), the left-hand side is a sum of four terms and the right-hand side contains a product of three variables. How many terms should you apply AM–GM to, and why does the 16 help?

- Pause and try** Now find the minimum of $x^5 + y^5 + z^5 - 5xyz$. (*Hint: you want a fifth root of $x^5 y^5 z^5$, so you need five numbers. What should the other two be?*)

Try this next

- The definition.** (From STEP II, 2018.) Draw a sketch illustrating the chord definition of concavity in the box above. Explain why a function with $f''(x) < 0$ satisfies it. Then, by choosing $t = \frac{1}{3}$, $x_1 = u$ and $x_2 = \frac{v+w}{2}$ suitably, show that a concave function satisfies Jensen's inequality for three points.
- A product of sines.** (From STEP II, 2018.) By considering $\ln(\sin x)$, show that if A, B, C are the angles of a triangle then $\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$.
- Acute triangles.** Show that $\tan x$ is convex on $(0, \frac{\pi}{2})$, and deduce that in an acute-angled triangle, $\tan A + \tan B + \tan C \geq 3\sqrt{3}$.

- 4 Power means.** Show that x^2 is convex, and deduce that $\left(\frac{a_1 + \dots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \dots + a_n^2}{n}$. Interpret this as a statement about the variance of a set of numbers.

Further reading

- **J. Michael Steele, *The Cauchy–Schwarz Master Class*** (Cambridge University Press). Chapter 6 is about convexity and Jensen’s inequality.
- **Claudi Alsina & Roger Nelsen, *When Less Is More: Visualizing Basic Inequalities*** (Mathematical Association of America). Inequalities proved by pictures, many of them using convexity.
- **Stephen Boyd & Lieven Vandenberghe, *Convex Optimization*** (Cambridge University Press). A modern classic, free online; the first chapters show why convexity matters in practice.

The full guided solution, and what the question leads to at university, is in Section 3.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/convexity.html