

3.2 AM–GM, Backwards

WORKSHEET

The question

Suppose that $a_i > 0$ for all $i > 0$. Show that

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2} \right)^2.$$

Prove by induction that for all positive integers m

$$a_1 \cdots a_{2^m} \leq \left(\frac{a_1 + \cdots + a_{2^m}}{2^m} \right)^{2^m}. \quad (*)$$

If $n < 2^m$, put $b_1 = a_1, b_2 = a_2, \dots, b_n = a_n$ and $b_{n+1} = \cdots = b_{2^m} = A$, where

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

By applying $(*)$ to the b_i , show that

$$a_1 \cdots a_n A^{(2^m - n)} \leq A^{2^m}$$

(notice that $b_1 + \cdots + b_n = nA$). Deduce the (arithmetic mean)/(geometric mean) inequality

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** Why might it be easier to go from 2^m numbers to 2^{m+1} numbers than from n to $n + 1$?

STAGE 2 Doubling

- ?** **Ask yourself** Split the 2^{m+1} numbers into a first half and a second half, and let P and Q be the arithmetic means of the two halves. What does the inductive hypothesis say about each half?

STAGE 3 Filling the gaps

- ?** **Ask yourself** What is the average of the padded list $b_1, \dots, b_{2^m}$?

- ✓** **Pause and try** Finish the proof.

Try this next

- 1 One step back.** Show directly that if AM–GM holds for n numbers, it holds for $n - 1$ numbers, by padding with one copy of the average.
- 2 An increasing sequence.** Apply AM–GM to the $n + 1$ numbers $1 + \frac{1}{n}, \dots, 1 + \frac{1}{n}, 1$ (with n copies of $1 + \frac{1}{n}$) to show that $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$.
- 3 The best box.** A closed rectangular box has surface area S . Use AM–GM on the areas of three different faces to show that its volume is at most $(S/6)^{3/2}$, with equality for a cube.
- 4 Harmonic mean.** By applying AM–GM to the numbers $1/a_i$, show that the harmonic mean $\frac{n}{1/a_1 + \dots + 1/a_n}$ is at most the geometric mean.

Further reading

- **J. Michael Steele, *The Cauchy–Schwarz Master Class*** (Cambridge University Press). A wonderful, problem-driven book on inequalities, starting from exactly this level. Chapter 2 is about AM–GM.

- **Hardy, Littlewood & Pólya, *Inequalities*** (Cambridge University Press). The classic reference, written by three of the great analysts of the twentieth century.
- **Arthur Engel, *Problem-Solving Strategies*** (Springer). Chapter 7 is a large collection of inequality problems from olympiads.

The full guided solution, and what the question leads to at university, is in Section 3.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/am-gm-backwards.html