

3.1 Spot the Flaw

WORKSHEET

The question

Criticise each step of the following arguments. You should correct the arguments where necessary and possible, and say (with justification) whether you think the conclusions are true even though the argument is incorrect.

- (i) The function g defined by

$$g(x) = \frac{2x^3 + 3}{x^4 + 4}$$

satisfies $g'(x) = 0$ only for $x = 0$ or $x = \pm 1$. Hence the stationary values are given by $x = 0$, $g(x) = \frac{3}{4}$ and $x = \pm 1$, $g(x) = 1$. Since $\frac{3}{4} < 1$, there is a minimum at $x = 0$ and maxima at $x = \pm 1$. Thus we must have $\frac{3}{4} \leq g(x) \leq 1$ for all x .

- (ii) $\int (1-x)^{-3} dx = -3(1-x)^{-4}$ and so $\int_{-1}^3 (1-x)^{-3} dx = 0$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself When you are asked to “criticise each step”, what should you check at each step? Make a list before you start.

STAGE 1 Is the derivative right?



Ask yourself Differentiate g with the quotient rule. Where is $g'(x) = 0$? Check $x = 1$ and $x = -1$ directly.

STAGE 3 Does comparing values tell you which is a maximum?



Ask yourself Suppose a function has exactly two stationary points, with values $\frac{3}{4}$ and 1. Does it follow that the first is a minimum and the second a maximum?

STAGE 4 Do stationary values bound a function?



Pause and try Even with correct stationary points, why can't you conclude that every value of g lies between the smallest and largest stationary values? Find a value of x with $g(x) < \frac{3}{4}$.

STAGE 5 The integral



Ask yourself Differentiate $-3(1-x)^{-4}$. Do you get $(1-x)^{-3}$?



Ask yourself With the correct antiderivative, $\left[\frac{1}{2}(1-x)^{-2}\right]_{-1}^3 = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = 0$. So the answer was right after all. Or was it?

Try this next

- 1 A substitution that lies.** (From STEP I, 1987.) Explain why the substitution $x = 1/t$ does not show that

$$\int_{-1}^1 \frac{dx}{(1+x^2)^2} \quad \text{and} \quad \int_{-1}^1 \frac{-t^2}{(1+t^2)^2} dt$$

are equal. Evaluate both integrals correctly. (One is positive and one is negative, so

they certainly differ.)

2 True or false? (From STEP I, 1998.) Justify your answers.

- (a) $a^{\ln b} = b^{\ln a}$ for all $a, b > 0$.
- (b) $\cos(\sin \theta) = \sin(\cos \theta)$ for all real θ .
- (c) There is a polynomial P such that $|P(\theta) - \cos \theta| \leq 10^{-6}$ for all real θ .
- (d) $x^4 + 3 + x^{-4} \geq 5$ for all $x > 0$.

3 All horses are the same colour. Find the flaw in this “proof” by induction: any one horse is the same colour as itself; given $n + 1$ horses, the first n are the same colour by the inductive hypothesis, and so are the last n ; the two groups overlap, so all $n + 1$ are the same colour.

4 An integral that does exist. Show that $\int_{-1}^1 |x|^{-1/2} dx$ exists, although the integrand is undefined at 0, and find its value.

Further reading

- **E. A. Maxwell, *Fallacies in Mathematics*** (Cambridge University Press). A classic collection of plausible wrong arguments, organised by topic, written for sixth-formers.
- **Bernard Gelbaum & John Olmsted, *Counterexamples in Analysis*** (Dover). Hundreds of examples showing why the hypotheses of theorems matter. Dip in once you have started an analysis course.
- **Imre Lakatos, *Proofs and Refutations*** (Cambridge University Press). A brilliant dialogue about how mathematicians find, and fix, errors in proofs.

The full guided solution, and what the question leads to at university, is in Section 3.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch03/spot-the-flaw.html