

2.9 Fractions That Close In on e

WORKSHEET

The question

- (i) Show that, for $n = 2, 3, 4, \dots$,

$$\frac{d^2}{dt^2} \left(t^n (1-t)^n \right) = nt^{n-2} (1-t)^{n-2} \left[(n-1) - 2(2n-1)t(1-t) \right].$$

- (ii) The sequence $T_0, T_1, \dots$ is defined by

$$T_n = \int_0^1 \frac{t^n (1-t)^n}{n!} e^t dt.$$

Show that, for $n \geq 2$,

$$T_n = T_{n-2} - 2(2n-1)T_{n-1}.$$

- (iii) Evaluate T_0 and T_1 and deduce that, for $n \geq 0$, T_n can be written in the form

$$T_n = a_n + b_n e,$$

where a_n and b_n are integers (which you should not attempt to evaluate).

- (iv) Show that $0 < T_n < \frac{e}{n!}$ for $n \geq 0$. Given that b_n is non-zero for all n , deduce that $\frac{-a_n}{b_n}$ tends to e as n tends to infinity.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What is the largest value of $t(1-t)$ on $[0, 1]$? What does that tell you about the size of T_n as n grows?

STAGE 1 Differentiating twice



Ask yourself Differentiate again. You will get two terms; take out the common factor $nt^{n-2}(1-t)^{n-2}$. To match the required form, you need to rewrite $(1-2t)^2$ in terms of $t(1-t)$. How?

STAGE 2 Integrating by parts twice



Ask yourself $p(t)$ is zero at both $t = 0$ and $t = 1$. What about $p'(t)$, for $n \geq 2$? Why does that matter when you integrate $p(t)e^t$ by parts?

STAGE 3 The integer structure



Ask yourself $T_0 = -1 + 1 \cdot e$ and $T_1 = 3 + (-1) \cdot e$. Why must every T_n have the form $a_n + b_n e$ with integer a_n, b_n ?

STAGE 4 Small, and therefore informative



Pause and try Divide $0 < a_n + b_n e < \frac{e}{n!}$ by $|b_n|$. What does that say about the distance between e and $-a_n/b_n$?

Try this next

- 1 Run the recurrence.** Compute a_n and b_n for $n = 2, 3, 4$ from the recurrence and check the fractions in the table. Show that the signs of b_n alternate.
- 2 A sharper bound.** Show that $0 < T_n < \frac{e}{4^n n!}$. How much does this improve the error estimate for $-a_n/b_n$?
- 3 The irrationality proof.** Write out the argument in the box above carefully. Where exactly do you use that $T_n > 0$?
- 4 A different weight.** **HARDER** Repeat the question with e^t replaced by e^{2t} (so $U_n =$

$\int_0^1 \frac{t^n(1-t)^n}{n!} e^{2t} dt$). Find a recurrence for U_n , show that U_n is a combination of 1 and e^2 with rational coefficients, and investigate whether the argument still proves that e^2 is irrational. (Compare Section 1.4.)

Further reading

- **Ivan Niven, *Irrational Numbers*** (Mathematical Association of America). The classic short book on irrationality proofs, including Hermite’s method.
- **Alfred van der Poorten, “A proof that Euler missed”, *The Mathematical Intelligencer* 1** (1979). A lively account of Apéry’s proof and its reception.
- **Julian Havil, *The Irrationals*** (Princeton University Press). A history of irrational numbers, written for a general mathematical reader.

The full guided solution, and what the question leads to at university, is in Section 2.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/e-approximations.html