# 2.8 Squaring and Subtracting Two

**WORKSHEET**

**The question**

A sequence $x_1, x_2, \ldots$ of real numbers is defined by $x_{n+1} = x_n^2 - 2$ for $n \geq 1$ and $x_1 = a$.

(i) Show that if $a > 2$ then $x_n \geq 2 + 4^{n-1}(a-2)$.

(ii) Show also that $x_n \to \infty$ as $n \to \infty$ if and only if $|a| > 2$.

(iii) When $a > 2$, a second sequence $y_1, y_2, \ldots$ is defined by

$$y_n = \frac{A x_1 x_2 \cdots x_n}{x_{n+1}},$$

where $A$ is a positive constant and $n \geq 1$.

Prove that, for a certain value of $a$, with $a > 2$, which you should find in terms of $A$,

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for all $n \geq 1$.

Determine whether, for this value of $a$, the second sequence converges.

## Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

**Ask yourself** Why can a number in $[-2, 2]$ never escape? What is the largest and smallest that $x^2 - 2$ can be when $-2 \le x \le 2$?

### STAGE 1 Escaping

**Ask yourself** Part (i) has a $4^{n-1}$ in it, suggesting that the distance above 2 at least quadruples at each step. Write $x_n = 2 + d$. What is $x_{n+1}$ in terms of $d$?

### STAGE 2 Escape if and only if $|a| > 2$

**Ask yourself** "If and only if" means two separate proofs. Which direction does part (i) almost give you, and what is missing?

### STAGE 3 A hidden formula

**Ask yourself** If $a > 2$, there is a number $t > 1$ with $a = t + \dfrac{1}{t}$. (Why?) What are $x_2$ and $x_3$ in terms of $t$?

### STAGE 4 The product telescopes

**Ask yourself** What is $\left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right)$? What happens if you multiply the result by $t^2 + \frac{1}{t^2}$?

**Pause and try** Compute $x_{n+1}^2 - 4$ in terms of $t$, and compare with the product.

## Try this next

1. **Inside the interval.** Prove that if $a = 2\cos\theta$ then $x_n = 2\cos(2^{n-1}\theta)$. Find all values of $a$ for which $x_3 = a$ but $x_2 \neq a$. (*Hint:* you want $4\theta \equiv \pm\theta$, modulo $2\pi$.)

2. **Lucas–Lehmer by hand.** Use the test to show that $M_5 = 31$ is prime and that $M_{11} = 2047$ is not. (For $M_{11}$ you will need nine steps; a calculator helps. Check your conclusion by factorising 2047.)

**3 An explicit example.** Take $A = \frac{3}{2}$. Find $a$ and $t$, and check the formula $y_1 = \sqrt{x_2^2 - 4}/x_2$ numerically.

**4 A sum that telescopes.** HARDER With $a = t + 1/t$ and $t > 1$, show that

$$\frac{1}{x_1 x_2 \cdots x_n} = \frac{t - t^{-1}}{t^{2^n} - t^{-2^n}}.$$

Using the identity $\dfrac{1}{z-1} - \dfrac{1}{z^2-1} = \dfrac{z}{z^2-1}$ with $z = t^{2^n}$, deduce that

$$\frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} + \cdots = \frac{1}{t}.$$

Check this numerically when $a = 3$.

## Further reading

- **Ian Stewart, *Does God Play Dice?*** (Penguin). The classic popular account of chaos, including the logistic map.
- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). The standard undergraduate introduction; Chapter 10 is about one-dimensional maps like this one.
- **Marcus du Sautoy, *The Music of the Primes*** (Fourth Estate). Includes the story of the hunt for Mersenne primes.

The full guided solution, and what the question leads to at university, is in Section 2.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/squaring-minus-two.html