

2.7 Six Digits or Six Hundred

WORKSHEET

The question

- (i) The sequence x_n for $n = 0, 1, 2, \dots$ is defined by $x_0 = 1$ and by

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

for $n \geq 0$.

- (a) Explain briefly why $x_n \geq 1$ for all n .
 (b) Show that $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, and that

$$|x_{n+1}^2 - 2| \leq \frac{1}{4} |x_n^2 - 2|.$$

- (c) Show that

$$2 - 10^{-6} \leq x_{10}^2 \leq 2.$$

- (ii) The sequence y_n for $n = 0, 1, 2, \dots$ is defined by $y_0 = 1$ and by

$$y_{n+1} = \frac{y_n^2 + 2}{2y_n}$$

for $n \geq 0$.

- (a) Show that, for $n \geq 0$,

$$y_{n+1} - \sqrt{2} = \frac{(y_n - \sqrt{2})^2}{2y_n}$$

and deduce that $y_n \geq 1$ for $n \geq 0$.

- (b) Show that

$$y_n - \sqrt{2} \leq 2 \left(\frac{\sqrt{2} - 1}{2} \right)^{2^n}$$

for $n \geq 1$.

- (c) Using the fact that $\sqrt{2} - 1 < \frac{1}{2}$, or otherwise, show that

$$\sqrt{2} \leq y_{10} \leq \sqrt{2} + 10^{-600}.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Do you notice anything about the two lists? (Look at which fractions appear in both.)

STAGE 2 The error in the first recipe



Ask yourself The question measures the error by $x_n^2 - 2$ rather than $x_n - \sqrt{2}$. Write $x_{n+1}^2 - 2$ as a single fraction in terms of x_n . What do you notice about the numerator?

STAGE 4 The error in the second recipe



Ask yourself Write $y_{n+1} - \sqrt{2}$ as a single fraction. The numerator should remind you of a perfect square.

STAGE 5 Squaring the error



Pause and try Prove by induction that $e_n \leq 2r^{2^n}$ for $n \geq 1$, where $r = \frac{\sqrt{2}-1}{2}$. Check the base case carefully.

Try this next

- Pell's equation.** Show that if $y_n = p/q$ in lowest terms with $p^2 - 2q^2 = 1$, then $y_{n+1} = \frac{p^2 + 2q^2}{2pq}$ also satisfies this equation. Check that $577^2 - 2 \times 408^2 = 1$.
- Cube roots.** Write down Newton's method for $f(y) = y^3 - 2$, and show that the new error is $\frac{(y_n - \alpha)^2(2y_n + \alpha)}{3y_n^2}$, where $\alpha = \sqrt[3]{2}$. Deduce that if $y_n > \alpha$ then the new error is at most $\frac{1}{\alpha}$ times the square of the old error. (*Hint:* use $\alpha^3 = 2$.)
- Division without dividing.** Show that Newton's method for $f(y) = \frac{1}{y} - a$ gives $y_{n+1} = y_n(2 - ay_n)$, which needs no division. If $e_n = 1 - ay_n$, show that $e_{n+1} = e_n^2$. (Computer processors use iterations like this to divide.)
- When Newton fails.** Apply Newton's method to $f(y) = y^{1/3}$, whose only root is 0. Show that each step doubles the distance from the root, whatever you start with. Which hypothesis of the error formula fails?

Further reading

- ▶ **Endre Süli & David Mayers, *An Introduction to Numerical Analysis*** (Cambridge University Press). The standard undergraduate text; Chapter 1 covers fixed-point iteration and Newton’s method.
- ▶ **Nick Higham (ed.), *The Princeton Companion to Applied Mathematics*** (Princeton University Press). Short articles on Newton’s method and numerical analysis, written by experts for a wide audience.
- ▶ **David Fowler & Eleanor Robson, “Square root approximations in Old Babylonian mathematics: YBC 7289 in context”, *Historia Mathematica* 25 (1998).**

The full guided solution, and what the question leads to at university, is in Section 2.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/newton-speed.html