

2.6 The Arithmetic–Geometric Mean

WORKSHEET

The question

In this question, the following theorem may be used without proof.

Let $u_1, u_2, \dots$ be a sequence of real numbers. If the sequence is

- ▶ bounded above, so $u_n \leq b$ for all n , where b is some fixed number
- ▶ and increasing, so $u_n \leq u_{n+1}$ for all n

then there is a number $L \leq b$ such that $u_n \rightarrow L$ as $n \rightarrow \infty$.

For positive real numbers x and y , define $a(x, y) = \frac{1}{2}(x + y)$ and $g(x, y) = \sqrt{xy}$. Let x_0 and y_0 be two positive real numbers with $y_0 < x_0$ and define, for $n \geq 0$,

$$x_{n+1} = a(x_n, y_n), \quad y_{n+1} = g(x_n, y_n).$$

- (i) By considering $(\sqrt{x_n} - \sqrt{y_n})^2$, show that $y_{n+1} < x_{n+1}$, for $n \geq 0$. Show further that, for $n \geq 0$,

- ▶ $x_{n+1} < x_n$
- ▶ $y_n < y_{n+1}$.

Deduce that there is a value M such that $y_n \rightarrow M$ as $n \rightarrow \infty$.

Show that $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$ and hence that $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$.

Explain why x_n also tends to M as $n \rightarrow \infty$.

- (ii) Let

$$I(p, q) = \int_0^\infty \frac{1}{\sqrt{(p^2 + x^2)(q^2 + x^2)}} dx,$$

where p and q are positive real numbers with $q < p$.

Show, using the substitution $t = \frac{1}{2} \left(x - \frac{pq}{x} \right)$ in the integral

$$\int_{-\infty}^\infty \frac{1}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)(pq + t^2)}} dt,$$

that

$$I(p, q) = I(a(p, q), g(p, q)).$$

Hence evaluate $I(x_0, y_0)$ in terms of M .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Why is the arithmetic mean of two different positive numbers always bigger than their geometric mean? Can you prove it in one line?

STAGE 1 The two sequences squeeze together



Ask yourself Why is the arithmetic mean of x_n and y_n less than x_n , and the geometric mean bigger than y_n ?

STAGE 2 The substitution



Ask yourself With $t = \frac{1}{2} \left(x - \frac{pq}{x} \right)$, compute $t^2 + pq$. Then compute $t^2 + \frac{1}{4}(p + q)^2$. Do either of them factorise nicely?



Pause and try Put these together. As x runs from 0 to ∞ , what does t do? What does the t -integral become?

STAGE 3 Evaluating the integral



Ask yourself Apply this identity repeatedly, starting from (x_0, y_0) . What is $I(x_0, y_0)$ equal to after n steps? What happens as $n \rightarrow \infty$?

Try this next

- Quadratic convergence.** Prove the identity $x_{n+1} - y_{n+1} = \frac{(x_n - y_n)^2}{2(\sqrt{x_n} + \sqrt{y_n})^2}$, and deduce that $x_{n+1} - y_{n+1} \leq \frac{(x_n - y_n)^2}{8y_0}$.
- By hand.** Compute $M(2, 1)$ to six decimal places using a calculator. How many steps did you need?
- Scaling.** Show that $M(\lambda x, \lambda y) = \lambda M(x, y)$ for $\lambda > 0$, and that $I(\lambda p, \lambda q) = \lambda^{-1} I(p, q)$. Check that these two facts are consistent with $I(x_0, y_0) = \pi/(2M)$.
- The pendulum.** Use the formula above to find, to three significant figures, how much longer the period of a pendulum released from 60° is than the small-angle prediction.

Further reading

- ▶ **David Cox**, “The arithmetic–geometric mean of Gauss”, *L’Enseignement Mathématique* 30 (1984). The classic account of the history and mathematics; the first sections need only calculus.
- ▶ **Jonathan Borwein & Peter Borwein**, *Pi and the AGM* (Wiley). The definitive book on the AGM and fast computation of π .
- ▶ **Jonathan Borwein & Keith Devlin**, *The Computer as Crucible* (A. K. Peters). Experimental mathematics, including discoveries like Gauss’s, made by spotting numerical coincidences.

The full guided solution, and what the question leads to at university, is in Section 2.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/agm.html