

2.5 Viète's Product

WORKSHEET

The question

You may assume that all infinite sums and products in this question converge.

- (i) Prove by induction that for all positive integers n ,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} \cdot \frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right).$$

- (ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as y tends to 0, $\frac{y}{\sinh y}$ tends to 1.

Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \cdots$$

- (iii) Let $x = \ln 2$. Evaluate $\cosh\left(\frac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1 + 2^{\frac{1}{2}}}{2 \times 2^{\frac{1}{4}}}.$$

Use part (ii) to show that

$$\frac{1}{\ln 2} = \frac{1 + 2^{\frac{1}{2}}}{2} \times \frac{1 + 2^{\frac{1}{4}}}{2} \times \frac{1 + 2^{\frac{1}{8}}}{2} \cdots$$

- (iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2 + \sqrt{2}}}{2} \times \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What is the double-angle formula for $\sinh$? (If you are not sure, write $\sinh 2y$ and $\sinh y \cosh y$ in terms of exponentials and compare.)

STAGE 1 Halving repeatedly



Ask yourself Suppose the formula holds for n . The last factor is $\sinh(x/2^n)$. How can you rewrite it to get the formula for $n + 1$?

STAGE 3 A formula for $\ln 2$



Ask yourself What is the general pattern for $\cosh(x/2^k)$? What happens when you multiply all the denominators $2 \times 2^{1/2^k}$ together?

STAGE 4 Viète's formula



Ask yourself Part (iv) has π in it, so it must come from ordinary trigonometry, not hyperbolic functions. Does everything in parts (i) and (ii) still work with $\sin$ and $\cos$ in place of $\sinh$ and $\cosh$?



Pause and try Choose x so that the left-hand side is $2/\pi$, and work out the first few factors. The half-angle formula $\cos \frac{\theta}{2} = \sqrt{\frac{1+\cos \theta}{2}}$ (for $0 \leq \theta \leq \pi$) will be useful.

Try this next

- 1 The 1990 version.** (From STEP I, 1990.) Show directly, using only the double-angle formula for $\sin$, that

$$\cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{4}\right) \cdots \cos\left(\frac{\alpha}{2^n}\right) = \frac{\sin \alpha}{2^n \sin(\alpha/2^n)},$$

and deduce Viète's formula in the form $\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2}} \cdots}$.

- 2 Another angle.** Use $x = \frac{\pi}{3}$ in the product for $\frac{\sin x}{x}$ to write $\frac{3\sqrt{3}}{2\pi}$ as an infinite product, and simplify the first two factors.

- 3 How many factors?** Use a calculator to find how many factors of Viète's product are needed to give π correct to 4 decimal places. Explain why each extra factor gains about 0.6 of a digit.
- 4 A formula for $\ln 3$.** Repeat part (iii) with $x = \ln 3$ to write $\frac{1}{\ln 3}$ as an infinite product. Check numerically that the first five factors give $\frac{1}{\ln 3}$ to two decimal places.

Further reading

- **Petr Beckmann, *A History of π*** (St Martin's Press). Archimedes, Viète, Wallis and the rest, told with gusto.
- **Jonathan Borwein & Peter Borwein, *Pi and the AGM*** (Wiley). The serious mathematics of computing π , including the method of Section 2.6. Chapter 1 is accessible with this chapter behind you.
- **Eli Maor, *Trigonometric Delights*** (Princeton University Press). Contains a chapter on Viète and his product.

The full guided solution, and what the question leads to at university, is in Section 2.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/viete.html