

2.4 $\pi^2/6$

WORKSHEET

The question

- (i) Use De Moivre's theorem to show that, if $\sin \theta \neq 0$, then

$$\frac{(\cot \theta + i)^{2n+1} - (\cot \theta - i)^{2n+1}}{2i} = \frac{\sin(2n+1)\theta}{\sin^{2n+1} \theta},$$

for any positive integer n .

Deduce that the solutions of the equation

$$\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \cdots + (-1)^n = 0$$

are

$$x = \cot^2 \left(\frac{m\pi}{2n+1} \right)$$

where $m = 1, 2, \dots, n$.

- (ii) Hence show that

$$\sum_{m=1}^n \cot^2 \left(\frac{m\pi}{2n+1} \right) = \frac{n(2n-1)}{3}.$$

- (iii) Given that $0 < \sin \theta < \theta < \tan \theta$ for $0 < \theta < \frac{1}{2}\pi$, show that

$$\cot^2 \theta < \frac{1}{\theta^2} < 1 + \cot^2 \theta.$$

Hence show that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** If a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ has roots $r_1, \dots, r_n$, what is $r_1 + \cdots + r_n$ in terms of the coefficients?

STAGE 1 De Moivre

? **Ask yourself** Write $\cot \theta + i$ as a single fraction. What do you notice about the numerator?

STAGE 2 The polynomial

? **Ask yourself** In $(c+i)^N - (c-i)^N$, which terms of the two binomial expansions cancel, and which double?

✎ **Pause and try** For which θ is the left-hand side zero? Use this to find n different roots of the polynomial.

STAGE 4 The squeeze

? **Ask yourself** Put $\theta = \frac{m\pi}{2n+1}$ and add up over $m = 1, \dots, n$. What do the two outer sums become? What does the middle sum have to do with $\sum 1/m^2$?

Try this next

1 The 1992 version. (From STEP III, 1992.) Using the polynomial from part (i), show that

$$\sum_{k=1}^n \tan^2 \left(\frac{k\pi}{2n+1} \right) = n(2n+1) \quad \text{and} \quad \sum_{k=1}^n \operatorname{cosec}^2 \left(\frac{k\pi}{2n+1} \right) = \frac{2n(n+1)}{3}.$$

(Hint for the first: the numbers $\tan^2$ are the reciprocals of the roots. What polynomial has roots $1/r_i$?)

2 Odd squares. Deduce from $\sum 1/m^2 = \pi^2/6$ that $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}$.

3 Fourth powers. **HARDER** Use $\sum r_i^2 = (\sum r_i)^2 - 2 \sum_{i < j} r_i r_j$ and the coefficients of the polynomial to find $\sum_{m=1}^n \cot^4 \left(\frac{m\pi}{2n+1} \right)$, and adapt the squeeze to prove $\sum 1/m^4 = \pi^4/90$.

- 4 How fast?** Using the lower bound in part (iii), show that $\frac{\pi^2}{6} - \sum_{m=1}^n \frac{1}{m^2} < \frac{\pi^2}{4n}$. How large an n does this guarantee is enough for the partial sum to be within 0.001 of $\pi^2/6$? Compare with the true error, which is close to $1/n$.

Further reading

- **William Dunham**, *Euler: The Master of Us All* (Mathematical Association of America). Chapter 3 tells the story of the Basel problem with the mathematics in full.
- **Aigner & Ziegler**, *Proofs from THE BOOK*, Chapter 9, “Three times $\pi^2/6$ ”. Includes the proof in this section.
- **Paul Nahin**, *Dr. Euler’s Fabulous Formula* (Princeton University Press). Complex numbers, Fourier series and sums like this one.

The full guided solution, and what the question leads to at university, is in Section 2.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/basel.html