

2.3 A Divergent Series and Its Convergent Cousin

WORKSHEET

The question

The numbers $f(r)$ satisfy $f(r) > f(r+1)$ for $r = 1, 2, \dots$. Show that, for any non-negative integer n ,

$$k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n),$$

where k is an integer greater than 1.

(i) By taking $f(r) = 1/r$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1.$$

Deduce that the sum $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $f(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\frac{1}{3}.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** The first inequality is about the block of terms from $r = k^n$ to $r = k^{n+1} - 1$. With $k = 2$ and $n = 3$, which terms are these, and how many of them are there?

STAGE 2 The harmonic series

? **Ask yourself** With $k = 2$ and $f(r) = 1/r$, what are the two bounds for the block from 2^n to $2^{n+1} - 1$?

STAGE 4 Counting numbers without a 2

? **Ask yourself** How many whole numbers from 0 to 999 can you write using only the digits 0, 1, 3, 4, 5, 6, 7, 8, 9, if you allow leading zeros (so 7 is written 007)?

STAGE 5 Kempner's series

✓ **Pause and try** How many d -digit numbers have no 2 in them? And what is the smallest a d -digit number can be?

Try this next

- 1 Feel the slowness.** Using $\sum_{r \leq N} 1/r \approx \ln N + 0.5772$, estimate how many terms of the harmonic series you need to pass 10, and to pass 100.
- 2 The test in action.** Use the method of part (i) with $f(r) = 1/r^2$ to show that $\sum 1/r^2 \leq 2$. Then improve the bound by treating the first term separately.
- 3 Between convergence and divergence.** Use the block inequality with $k = 2$ to show that $\sum_{r \geq 2} \frac{1}{r \ln r}$ diverges and $\sum_{r \geq 2} \frac{1}{r(\ln r)^2}$ converges.
- 4 Excluding more.** **HARDER** Show that the sum of the reciprocals of the positive integers with no 2 and no 7 in their decimal representation is less than 35. (Can you see where 35 comes from?)

Further reading

- ▶ **Julian Havil**, *Gamma: Exploring Euler's Constant* (Princeton University Press). The harmonic series, γ and much more, written for a strong school student.
- ▶ **Mary Hart**, *Guide to Analysis* (Palgrave). A concise first-year text with a clear chapter on series and comparison tests.
- ▶ **Robert Baillie**, “Sums of reciprocals of integers missing a given digit”, *American Mathematical Monthly* 86 (1979). A short paper you can read with this section behind you.

The full guided solution, and what the question leads to at university, is in Section 2.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/kempner.html