

2.2 The Golden Ratio as a Limit

WORKSHEET

The question

In this question, the following theorem may be used.

Let $u_1, u_2, \dots$ be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leq b$ for all n , where b is some fixed number) and increasing (that is, $u_n \geq u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges).

The sequence $u_1, u_2, \dots$ is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \quad (n \geq 1). \quad (*)$$

- (i) Show that, for $n \geq 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}.$$

- (ii) Prove, by induction or otherwise, that $1 \leq u_n \leq 2$ for all n .
- (iii) Show that the sequence $u_1, u_3, u_5, \dots$ tends to a limit, and that the sequence $u_2, u_4, u_6, \dots$ tends to a limit. Find these limits and deduce that the sequence $u_1, u_2, u_3, \dots$ tends to a limit.

Would this conclusion change if the sequence were defined by $(*)$ and $u_1 = 3$?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Do you recognise the numerators and denominators? And look at the decimals: 1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, ... Is the sequence increasing? Decreasing?

STAGE 1 Comparing terms two apart

Ask yourself Using (*) twice, write $u_{n+2} - u_n$ in terms of $u_{n+1} - u_{n-1}$, and then $u_{n+1} - u_{n-1}$ in terms of $u_n - u_{n-2}$.

STAGE 2 A trap

Pause and try Prove that $1 \leq u_n \leq 2$ for all n .

STAGE 3 Two monotone subsequences

Ask yourself The formula in part (i) says that $u_{n+2} - u_n$ is a positive multiple of $u_n - u_{n-2}$. What does that tell you about the signs of $u_3 - u_1$, $u_5 - u_3$, $u_7 - u_5$, ...?

STAGE 4 Finding the limits

Ask yourself To find L , you need an equation relating u_{n+2} to u_n directly. Can you get one from (*)?

Ask yourself The odd terms tend to φ and the even terms tend to φ . Why does this mean that the whole sequence tends to φ ?

Try this next

- Fibonacci inside.** Prove that $u_n = F_{n+1}/F_n$, where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
- The error shrinks.** Show that $u_{n+1} - \varphi = -\frac{u_n - \varphi}{\varphi u_n}$, and deduce that $|u_n - \varphi| \leq \varphi^{-(n-1)}$ for all n . (Use $\varphi^2 = \varphi + 1$, and $u_n \geq 1$.)
- Breaking the sequence.** Show that if $u_1 = \frac{1-\sqrt{5}}{2}$ then the sequence is constant. Find

all the starting values u_1 for which some $u_n = 0$, so that the sequence cannot continue. (*Hint*: work backwards from $u_n = 0$.)

- 4 **A cousin for $\sqrt{2}$.** Let $v_1 = 1$ and $v_{n+1} = 1 + \frac{1}{1+v_n}$. Adapt the method of this question to show that $v_n \rightarrow \sqrt{2}$, and write $\sqrt{2}$ as a continued fraction.

Further reading

- **Mario Livio, *The Golden Ratio*** (Broadway). The history of φ , separating the genuine mathematics from the many myths.
- **C. D. Olds, *Continued Fractions*** (Mathematical Association of America). A short, very readable introduction aimed at strong school students.
- **J. C. Burkill, *A First Course in Mathematical Analysis*** (Cambridge University Press). Rigorous treatment of monotone sequences and completeness, with many exercises.

The full guided solution, and what the question leads to at university, is in Section 2.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/golden-limit.html