

2.1 What "Eventually" Means

WORKSHEET

The question

If $s_1, s_2, s_3, \dots$ and $t_1, t_2, t_3, \dots$ are sequences of positive numbers, we write

$$(s_n) \leq (t_n)$$

to mean

“there exists a positive integer m such that $s_n \leq t_n$ whenever $n \geq m$ ”.

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m ; in the case of a false statement, you should give a counterexample.

- (i) $(1000n) \leq (n^2)$.
- (ii) If it is not the case that $(s_n) \leq (t_n)$, then it is the case that $(t_n) \leq (s_n)$.
- (iii) If $(s_n) \leq (t_n)$ and $(t_n) \leq (u_n)$, then $(s_n) \leq (u_n)$.
- (iv) $(n^2) \leq (2^n)$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** The definition contains the phrase “there exists a positive integer m ”. To prove that $(s_n) \leq (t_n)$, what exactly do you have to produce? To prove that it is *false*, what do you have to show?

STAGE 2 Is “eventually smaller” like ordinary “smaller”?

- ?** **Ask yourself** What does “it is *not* the case that $(s_n) \leq (t_n)$ ” actually mean? Write it out in full, without the word “not” in front.

- /** **Pause and try** Find two sequences of positive numbers that swap places for ever, so that neither is eventually smaller than the other.

STAGE 3 Chaining

- ?** **Ask yourself** In part (iii) you are given two values of m , one for each hypothesis. They might be different. Which one should you use for the conclusion?

STAGE 4 A quadratic against an exponential

- ?** **Ask yourself** To prove $n^2 \leq 2^n$ for all $n \geq 4$ by induction, you need to get from $n^2 \leq 2^n$ to $(n+1)^2 \leq 2^{n+1} = 2 \cdot 2^n$. What inequality between $(n+1)^2$ and n^2 would make that step work?

Try this next

- 1 The specimen warm-up.** (From the STEP III Specimen paper, issued before the first STEP in 1987.) Show by using the binomial expansion that $(1+x)^n > nx$ whenever $x \geq 0$ and n is a positive integer. Deduce that if $y > 1$ then, given any number K , an integer N can be found such that $y^n > K$ for all $n \geq N$. Show further that if $y > 1$ then, given any K , an integer N can be found such that $y^n/n > K$ for all $n \geq N$. (*Hint for the last part:* use the next term of the binomial expansion.)
- 2 Both ways round.** Prove that if $(s_n) \leq (t_n)$ and $(t_n) \leq (s_n)$, then $s_n = t_n$ for all sufficiently large n .
- 3 Racing to infinity.** Find two *increasing* sequences of positive numbers, both tending

to infinity, neither of which is eventually smaller than the other.

- 4 Up the ladder.** Show that $(2^n) \leq (n!)$, giving an explicit m . Then show that $(n^{10}) \leq (2^n)$ and find the smallest m that works. (A spreadsheet helps for the second part; the answer is surprisingly large.)

Further reading

- **Lara Alcock**, *How to Study for a Mathematics Degree* (Oxford University Press). Chapter 3 is about exactly this kind of logical language and why new undergraduates find it hard.
- **Lara Alcock**, *How to Think About Analysis* (Oxford University Press). A friendly guide to the definitions of first-year analysis.
- **Daniel Velleman**, *How to Prove It* (Cambridge University Press). A systematic introduction to logic, quantifiers and proof.

The full guided solution, and what the question leads to at university, is in Section 2.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch02/eventually.html