

13.7 Manufacturing Correlation

WORKSHEET

The question

In this question, $\text{Corr}(U, V)$ denotes the product moment correlation coefficient between the random variables U and V , defined by

$$\text{Corr}(U, V) \equiv \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\text{Var}(V)}}.$$

The independent random variables Z_1 , Z_2 and Z_3 each have expectation 0 and variance 1. What is the value of $\text{Corr}(Z_1, Z_2)$?

Let $Y_1 = Z_1$ and let

$$Y_2 = \rho_{12}Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}}Z_2,$$

where ρ_{12} is a given constant with $-1 < \rho_{12} < 1$. Find $E(Y_2)$, $\text{Var}(Y_2)$ and $\text{Corr}(Y_1, Y_2)$.

Now let $Y_3 = aZ_1 + bZ_2 + cZ_3$, where a , b and c are real constants and $c \geq 0$. Given that $E(Y_3) = 0$, $\text{Var}(Y_3) = 1$, $\text{Corr}(Y_1, Y_3) = \rho_{13}$ and $\text{Corr}(Y_2, Y_3) = \rho_{23}$, express a , b and c in terms of ρ_{23} , ρ_{13} and ρ_{12} .

Given constants μ_i and σ_i , for $i = 1, 2$ and 3 , give expressions in terms of the Y_i for random variables X_i such that $E(X_i) = \mu_i$, $\text{Var}(X_i) = \sigma_i^2$ and $\text{Corr}(X_i, X_j) = \rho_{ij}$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Z_1 and Z_2 are independent. What is their covariance, and so their correlation?

STAGE 1 Two variables



Ask yourself Think of Z_1 and Z_2 as perpendicular unit vectors. Where is Y_2 ? What is the cosine of the angle between Y_1 and Y_2 ?

STAGE 2 The third variable



Ask yourself Write down the three conditions $\text{Var}(Y_3) = 1$, $\text{Cov}(Y_1, Y_3) = \rho_{13}$, $\text{Cov}(Y_2, Y_3) = \rho_{23}$ in terms of a, b, c .

STAGE 3 Any means and variances



Pause and try The formula for c needs the quantity under the square root to be non-negative. Try $\rho_{12} = \rho_{13} = \rho_{23} = -0.6$. What goes wrong, and why does it make sense that three variables cannot all be strongly negatively correlated with each other?

Try this next

- Impossible correlations.** Show that three random variables cannot all have pairwise correlation ρ unless $\rho \geq -\frac{1}{2}$. What is the corresponding statement for n variables?
- Check the determinant.** Show that $1 - \rho_{13}^2 - b^2 \geq 0$ is equivalent to $\det R \geq 0$, where R is the 3×3 correlation matrix.
- Portfolio risk.** Two shares have returns with standard deviations $\sigma_1 = 0.2$ and $\sigma_2 = 0.3$ and correlation ρ . Find the standard deviation of a portfolio half in each, for $\rho = -1, 0$ and 1 . What does diversification achieve?
- Recover the ingredients.** Given Y_1, Y_2 as in the question, express Z_2 in terms of Y_1 and Y_2 . (This is the Gram–Schmidt step.)

Further reading

- **Gilbert Strang, *Introduction to Linear Algebra*** (Wellesley-Cambridge), on positive defi-

nite matrices and the Cholesky factorisation.

- ▶ **Paul Glasserman**, *Monte Carlo Methods in Financial Engineering* (Springer). Simulating correlated variables in practice; advanced but the early chapters are readable.
- ▶ **Theodore Porter**, *Karl Pearson: The Scientific Life in a Statistical Age* (Princeton University Press).

The full guided solution, and what the question leads to at university, is in Section 13.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch13/correlation.html