

13.6 Errors That Add Up

WORKSHEET

The question

Two computers, LEP and VOZ are programmed to add numbers after first approximating each number by an integer. LEP approximates the numbers by rounding: that is, it replaces each number by the nearest integer. VOZ approximates by truncation: that is, it replaces each number by the largest integer less than or equal to the number. The fractional parts of the numbers to be added are uniformly and independently distributed. (The fractional part of a number a is $a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ is the largest integer less than or equal to a .) Both computers approximate and add 1500 numbers. For each computer, find the probability that the magnitude of error in the answer will exceed 15.

How many additions can LEP perform before the probability that the magnitude of error is less than 10 drops below 0.9?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** If a number's fractional part U is uniform on $[0, 1)$, what is the error when it is truncated? When it is rounded? What are the mean and variance of each?

STAGE 2 The probability of a large error

? **Ask yourself** For LEP, what is $P(|E| > 15)$ with $E \sim N(0, 125)$? For VOZ, where is the error almost certainly?

STAGE 3 How many additions?

? **Ask yourself** After n additions, LEP's error has standard deviation $\sqrt{n/12}$. When is $P(|E| < 10) \geq 0.9$?

Try this next

- 1 **VOZ corrected.** If VOZ adds $\frac{n}{2}$ to its answer at the end, what is the distribution of its error? Is it now as good as LEP?
- 2 **Rounding to one decimal place.** If numbers are rounded to one decimal place, with uniformly distributed discarded digits, find the standard deviation of the total error in adding 10 000 numbers.
- 3 **The Vancouver index.** If the index was recalculated 3000 times a day and each truncation lost on average 0.0005, how much would it lose in a year of 250 trading days?
- 4 **The exact distribution.** For the sum of two rounding errors, find the exact density (a triangle). For three, sketch it. How quickly does it start to look normal?

Further reading

- **Nicholas Higham, *Accuracy and Stability of Numerical Algorithms*** (SIAM). The authoritative book on rounding errors; the opening chapter is accessible.
- **David Goldberg, "What every computer scientist should know about floating-point arithmetic", *ACM Computing Surveys* 23 (1991).** A classic, free online.
- **Tim Harford, *How to Make the World Add Up*** (Bridge Street), on systematic errors in

data.

The full guided solution, and what the question leads to at university, is in Section 13.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch13/errors-add-up.html