

12.9 A Fair Decision from a Biased Coin

WORKSHEET

The question

- (i) Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the n th round is

$$(q^2 + p^2)(2qp)^{n-1}.$$

Show also that the probability that they make a decision on or before the n th round is at least

$$1 - \frac{1}{2^n}$$

whatever the value of p .

- (ii) Three people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In the first round of the procedure, all three people toss their coins once. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they continue to the next round.

In the next round the two people whose coins showed the same face toss again, but the third person just turns over his or her coin. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they go to the third round.

Show that the probability that they make a decision on or before the second round is at least $\frac{7}{16}$, whatever the value of p .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself In one round, what is the probability of a decision? Of no decision? What is the largest value $2pq$ can take?

STAGE 2 Decision by round n



Ask yourself What is the probability of *no* decision in the first n rounds?

STAGE 3 Three people



Ask yourself After a first round with two heads and one tail, the third person turns the tail over to a head. What must happen in round two for a decision? And after two tails and one head?



Ask yourself Write everything in terms of $s = pq$. (Use $p + q = 1$.)

Try this next

- 1 **Expected rounds.** Find the expected number of rounds for the procedure in part (i), and for von Neumann's procedure. For which p is von Neumann's procedure slower?
- 2 **Fair three-way choice.** Using a biased coin, devise a procedure that chooses one of three options, each with probability exactly $\frac{1}{3}$.
- 3 **A biased die from a fair coin.** Using a fair coin, devise a procedure that simulates a coin with probability of heads exactly $\frac{1}{3}$. What is the expected number of tosses?
- 4 **Three people again.** In part (ii), what is the probability that they go to the hotel, as a function of p ? Is the procedure fair when $p = \frac{1}{2}$?

Further reading

- John von Neumann, "Various techniques used in connection with random digits", *National Bureau of Standards Applied Mathematics Series* 12 (1951). Short and famous.
- Donald Knuth, *The Art of Computer Programming, Vol. 2* (Addison-Wesley), Chapter 3, on random numbers.
- Michael Mitzenmacher & Eli Upfal, *Probability and Computing* (Cambridge University

Press), on extracting randomness and entropy.

The full guided solution, and what the question leads to at university, is in Section 12.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/von-neumann.html