

12.8 Penney's Game

WORKSHEET

The question

In a game for two players, a fair coin is tossed repeatedly. Each player is assigned a sequence of heads and tails and the player whose sequence appears first wins. Four players, A , B , C and D take turns to play the game. Each time they play, A is assigned the sequence TTH (i.e. Tail then Tail then Head), B is assigned THH, C is assigned HHT and D is assigned HTT.

- (i) A and B play the game. Let p_{HH} , p_{HT} , p_{TH} and p_{TT} be the probabilities of A winning the game given that the first two tosses of the coin show HH, HT, TH and TT, respectively. Explain why $p_{TT} = 1$, and why $p_{HT} = \frac{1}{2}p_{TH} + \frac{1}{2}p_{TT}$. Show that $p_{HH} = p_{HT} = \frac{2}{3}$ and that $p_{TH} = \frac{1}{3}$. Deduce that the probability that A wins the game is $\frac{2}{3}$.
- (ii) B and C play the game. Find the probability that B wins.
- (iii) Show that if C plays D , then C is more likely to win than D , but that if D plays A , then D is more likely to win than A .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Suppose the tosses so far end in TT and nobody has won yet. Can THH ever appear before TTH?

STAGE 1 States



Ask yourself Neither sequence can be completed in two tosses. After two tosses, what is the next toss's effect from each state?

STAGE 3 *B* against *C*



Ask yourself *C* wants HHT and *B* wants THH. When does *C* win? (Look at what must come just before the first HH.)

Try this next

- 1 **Four players together.** (From STEP II, 2015.) Players choose HHT, THH, TTH and HTT. Show that if only HHT and THH play, HHT wins with probability $\frac{1}{4}$. If all four play together, show that each wins with probability $\frac{1}{4}$. If only THH and TTH play, find the probability that TTH wins.
- 2 **Waiting times.** Show that the expected number of tosses until HH first appears is 6, and until HT is 4. (Use first-step analysis.)
- 3 **The best reply.** For each of the eight sequences of length three, find the sequence that beats it with the greatest probability. Which sequence is beaten most easily?
- 4 **Conway's formula.** Use Conway's formula to check that HHT beats HTT with probability $\frac{2}{3}$.

Further reading

- **Martin Gardner, *Time Travel and Other Mathematical Bewilderments*** (W.H. Freeman). Includes Penney's game and Conway's formula.
- **Ronald Graham, Donald Knuth & Oren Patashnik, *Concrete Mathematics*** (Addison-Wesley), Section 8.4. Penney's game with generating functions.
- **David Williams, *Probability with Martingales*** (Cambridge University Press). The martingale proof, for later.

The full guided solution, and what the question leads to at university, is in Section 12.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/penney.html