

12.7 Who Meets Whom in a Knockout

WORKSHEET

The question

A tennis tournament is arranged for 2^n players. It is organised as a knockout tournament so that only the winners in any given round proceed to the next round. Opponents in each round except the final are drawn at random, and in any match either player has a probability of $\frac{1}{2}$ of winning. Two players are chosen at random before the draw for the first round. Find the probabilities that they play each other:

- (i) in the first round,
- (ii) in the final,
- (iii) in the tournament.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself How many matches are played in a knockout tournament with 2^n players? (Each match eliminates one player.)

STAGE 2 The key symmetry



Ask yourself The draw in each round is random and every match is a fair coin toss. Does the process treat any pair of players differently from any other?

STAGE 4 At any stage



Ask yourself X and Y can meet at most once. How does that let you add probabilities over matches?

Try this next

- 1 **Round by round.** Show directly that the probability that X and Y meet in round k is $\frac{2^{n-k}}{\binom{2^n}{2}}$, and check that these add up to $\frac{1}{2^{n-1}}$.
- 2 **A fixed bracket.** If instead the bracket is fixed in advance (as in the figure) but all players are placed in it at random, does the answer change?
- 3 **A strong player.** Suppose player X wins every match. What is now the probability that X meets Y ?
- 4 **A league.** In a league of $2n$ teams where every pair meets once, how many matches are there? What is the analogous “probability of meeting”?

Further reading

- ▶ **Frederick Mosteller**, *Fifty Challenging Problems in Probability* (Dover), which includes tournament problems.
- ▶ **David Spiegelhalter**, *The Art of Statistics* (Pelican), on modelling and forecasting.
- ▶ **Joseph Blitzstein & Jessica Hwang**, *Introduction to Probability* (CRC Press; free online), on indicators and symmetry.

The full guided solution, and what the question leads to at university, is in Section 12.7:

https://keithmcnulty.github.io/step_book_mcnulty/ch12/knockout.html