

12.6 One More Coin

WORKSHEET

The question

- (i) Alice tosses a fair coin twice and Bob tosses a fair coin three times. Calculate the probability that Bob gets more heads than Alice.
- (ii) Alice tosses a fair coin three times and Bob tosses a fair coin four times. Calculate the probability that Bob gets more heads than Alice.
- (iii) Let p_1 be the probability that Bob gets the same number of heads as Alice, and let p_2 be the probability that Bob gets more heads than Alice, when Alice and Bob each toss a fair coin n times.

Alice tosses a fair coin n times and Bob tosses a fair coin $n + 1$ times. Express the probability that Bob gets more heads than Alice in terms of p_1 and p_2 , and hence obtain a generalisation of the results of parts (i) and (ii).

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Alice tosses twice: what is the distribution of her number of heads? And Bob, tossing three times?

STAGE 2 Splitting Bob's tosses



Ask yourself Think of Bob's $n + 1$ tosses as n tosses followed by one extra toss. After his first n tosses, he is ahead of Alice, level with her, or behind. What happens in each case?



Pause and try Here is the one-sentence proof. Bob has $n + 1$ tosses, so his heads and tails add to $n + 1$, one more than Alice's. Show that Bob has more heads than Alice if and only if he does *not* have more tails than Alice. Since heads and tails are symmetric, these two events have equal probability, so each is $\frac{1}{2}$.

Try this next

- Two extra coins.** If Bob tosses $n + 2$ coins and Alice n , prove $P(\text{Bob more heads}) = \frac{1}{2} + \frac{1}{2}P(H' = H_A)$, where H' is the number of heads in Bob's first $n + 1$ tosses.
- Biased coins.** If both use a coin with probability p of heads, is the answer still $\frac{1}{2}$ with one extra toss? Which step of the slick proof fails?
- Ties.** Show that $p_1 = \binom{2n}{n}/4^n$ by the trick in the text, and evaluate it for $n = 1, 2, 3$.
- Dice.** Alice rolls n dice and Bob rolls $n + 1$. Is the probability that Bob gets more sixes equal to $\frac{1}{2}$?

Further reading

- **Peter Winkler, *Mathematical Mind-Benders*** (A K Peters). Many probability puzzles with symmetric solutions.
- **Gabor Székely, *Paradoxes in Probability Theory and Mathematical Statistics*** (Reidel).
- **Frederick Mosteller, *Fifty Challenging Problems in Probability*** (Dover).

The full guided solution, and what the question leads to at university, is in Section 12.6:

https://keithmcnulty.github.io/step_book_mcnulty/ch12/one-more-coin.html