

12.5 The Last Passenger's Seat

WORKSHEET

The question

A train has n seats, where $n \geq 2$. For a particular journey, all n seats have been sold, and each of the n passengers has been allocated a seat. The passengers arrive one at a time and are labelled $T_1, \dots, T_n$ according to the order in which they arrive: T_1 arrives first and T_n arrives last. The seat allocated to T_r ($r = 1, \dots, n$) is labelled S_r .

Passenger T_1 ignores their allocation and decides to choose a seat at random (each of the n seats being equally likely). However, for each $r \geq 2$, passenger T_r sits in S_r if it is available or, if S_r is not available, chooses from the available seats at random.

- (i) Let P_n be the probability that, in a train with n seats, T_n sits in S_n . Write down the value of P_2 and find the value of P_3 .
- (ii) Explain why, for $k = 2, 3, \dots, n-1$,

$$P(T_n \text{ sits in } S_n \mid T_1 \text{ sits in } S_k) = P_{n-k+1},$$

and deduce that, for $n \geq 3$,

$$P_n = \frac{1}{n} \left(1 + \sum_{r=2}^{n-1} P_r \right).$$

- (iii) Give the value of P_n in its simplest form and prove your result by induction.
- (iv) Let Q_n be the probability that, in a train with n seats, T_{n-1} sits in S_{n-1} . Determine Q_n for $n \geq 2$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

 **Pause and try** Work out P_2 and P_3 by listing what can happen.

STAGE 1 The recurrence

 **Ask yourself** Suppose T_1 sits in S_k , with $2 \leq k \leq n-1$. What happens to passengers $T_2, \dots, T_{k-1}$? When T_k arrives, which seats are free, and how does the situation compare with a smaller train?

STAGE 2 The answer

 **Ask yourself** $P_2 = P_3 = \frac{1}{2}$. Guess P_n and prove it by strong induction.

STAGE 3 The penultimate passenger

 **Ask yourself** Try to find a slicker argument. At the end, which seat is left for T_n ? Could it be S_2 ?

Try this next

- Two forgetful passengers.** If the first *two* passengers both choose at random, what is the probability that the last gets their own seat?
- Displaced passengers.** Verify the expected number of displaced passengers for $n = 3$ by listing the possibilities.
- An indirect route.** Show, using the recurrence, that $\sum_{r=2}^{n-1} P_r = \frac{n-2}{2}$ is consistent with the symmetry argument.
- The last seat.** Show that, conditional on the last passenger not getting their own seat, the last passenger gets S_1 with probability 1. Why?

Further reading

- **Peter Winkler, *Mathematical Puzzles: A Connoisseur's Collection*** (A K Peters). Includes the airline seat puzzle and many others solved by symmetry.
- **Frederick Mosteller, *Fifty Challenging Problems in Probability*** (Dover). A classic collection.

- **Joseph Blitzstein & Jessica Hwang, *Introduction to Probability*** (CRC Press; free online).

The full guided solution, and what the question leads to at university, is in Section 12.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/airline-seat.html