

12.4 Independent in Pairs, Not All Together

WORKSHEET

The question

- (i) In a game, each member of a team of n players rolls a fair six-sided die. The total score of the team is the number of pairs of players rolling the same number. For example, if 7 players roll 3, 3, 3, 3, 6, 6, 2 the total score is 7, as six different pairs of players both score 3 and one pair of players both score 6.

Let X_{ij} , for $1 \leq i < j \leq n$, be the random variable that takes the value 1 if players i and j roll the same number and the value 0 otherwise. Show that X_{12} is independent of X_{23} . Hence find the mean and variance of the team's total score.

- (ii) Show that, if Y_i , for $1 \leq i \leq m$, are random variables with mean zero, then

$$\text{Var}(Y_1 + Y_2 + \cdots + Y_m) = \sum_{i=1}^m E(Y_i^2) + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m E(Y_i Y_j).$$

- (iii) In a different game, each member of a team of n players rolls a fair six-sided die. The total score of the team is the number of pairs of players rolling the same even number minus the number of pairs of players rolling the same odd number. For example, if 7 players roll 3, 3, 3, 3, 6, 6, 2 the total score is -5 .

Let Z_{ij} , for $1 \leq i < j \leq n$, be the random variable that takes the value 1 if players i and j roll the same even number, the value -1 if players i and j roll the same odd number and the value 0 otherwise. Show that Z_{12} is not independent of Z_{23} . Find the mean of the team's total score and show that the variance of the team's total score is $\frac{1}{36}n(n^2 - 1)$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** What is $P(X_{12} = 1)$? What is $P(X_{12} = 1 \text{ and } X_{23} = 1)$?

STAGE 2 Mean and variance of the score

? **Ask yourself** When does the variance of a sum equal the sum of the variances?

STAGE 4 The odd-and-even game

? **Ask yourself** What is $E(Z_{12})$? What values can $Z_{12}Z_{23}$ take, and when is it non-zero?

Try this next

- 1 Two values.** (From STEP III, 2000.) X takes the values $x_1 \neq x_2$ and Y the values $y_1 \neq y_2$, with $P(X = x_1, Y = y_1) = a$, $P(X = x_1) = q$, $P(Y = y_1) = p$. Show that if $E(XY) = E(X)E(Y)$ then $(a - pq)(x_1 - x_2)(y_1 - y_2) = 0$, and deduce that X and Y are independent. Give a joint distribution for A and B , each uniform on $\{-1, 0, 1\}$, with $E(AB) = E(A)E(B)$ but not independent.
- 2 A bigger team.** For $n = 7$ in part (i), find the mean and standard deviation of the score, and compare the example score 7 with them.
- 3 Birthdays again.** Use indicators to find the mean and variance of the number of pairs sharing a birthday among k people (Section 9.9). Are the indicators pairwise independent?
- 4 Pairwise but not triple.** Find three events, each of probability $\frac{1}{2}$, with $P(A \cap B \cap C) = \frac{1}{8}$ but which are not pairwise independent.

Further reading

- **Geoffrey Grimmett & David Stirzaker, *Probability and Random Processes*** (Oxford University Press), the opening chapters on independence.
- **Michael Mitzenmacher & Eli Upfal, *Probability and Computing*** (Cambridge University Press). Pairwise independence and its uses in algorithms.
- **Joseph Blitzstein & Jessica Hwang, *Introduction to Probability*** (CRC Press; free online), on indicators and covariance.

The full guided solution, and what the question leads to at university, is in Section 12.4:

https://keithmcnulty.github.io/step_book_mcnulty/ch12/pairwise-independence.html