

12.3 What Does "Random" Mean?

WORKSHEET

The question

- (i) The point A lies on the circumference of a circle of radius a and centre O . The point B is chosen at random on the circumference, so that the angle AOB has a uniform distribution on $[0, 2\pi]$. Find the expected length of the chord AB .
- (ii) The point C is chosen at random in the interior of a circle of radius a and centre O , so that the probability that it lies in any given region is proportional to the area of the region. The random variable R is defined as the distance between C and O . Find the probability density function of R .

Obtain a formula in terms of a , R and t for the length of a chord through C that makes an acute angle of t with OC .

Show that as C varies (with t fixed), the expected length $L(t)$ of such chords is given by

$$L(t) = \frac{4a(1 - \cos^3 t)}{3 \sin^2 t}.$$

Show further that

$$L(t) = \frac{4a}{3} \left(\cos t + \frac{1}{2} \sec^2\left(\frac{1}{2}t\right) \right).$$

- (iii) The random variable T is uniformly distributed on $[0, \frac{1}{2}\pi]$. Find the expected value of $L(T)$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself If the angle AOB is θ , what is the length of the chord AB ? (Drop a perpendicular from O .)

STAGE 2 A random point inside



Ask yourself What is $P(R \leq r)$ for $0 \leq r \leq a$? Differentiate to get the density.



Ask yourself A chord through C makes angle t with OC . How far is the chord from O ?

STAGE 3 Averaging over C



Ask yourself Factorise $1 - \cos^3 t$ and $\sin^2 t = 1 - \cos^2 t$. What cancels?

Try this next

- Bertrand's probabilities.** Verify the three probabilities $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$ above.
- The other mean.** Show that if the chord's midpoint is chosen uniformly in the disc (the right-hand picture), the expected length is $\frac{4a}{3}$. (*Hint*: this is $L(\frac{\pi}{2})$. Why?)
- Random distance.** If the distance of the chord from O is uniform on $[0, a]$, show that the expected length is $\frac{\pi a}{2}$.
- Three methods, three distributions.** For each of the three methods, find the probability density of the chord's length ℓ on $[0, 2a]$.

Further reading

- **Edwin Jaynes**, “The well-posed problem”, *Foundations of Physics* 3 (1973). The invariance argument, readable and provocative.
- **Gábor Székely**, *Paradoxes in Probability Theory and Mathematical Statistics* (Reidel). Bertrand's paradox and many others.
- **Daniel Klain & Gian-Carlo Rota**, *Introduction to Geometric Probability* (Cambridge University Press). Integral geometry, for later.

The full guided solution, and what the question leads to at university, is in Section 12.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/bertrand.html