12.24 The Central Limit Theorem

WORKSHEET

The question

Let X be a random variable with a Laplace distribution, so that its probability density function is given by

$$f(x) = \tfrac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty. \qquad (*)$$

Sketch $f(x)$. Show that its moment generating function $M_X(\theta)$ is given by $M_X(\theta) = (1-\theta^2)^{-1}$ and hence find the variance of X.

A frog is jumping up and down, attempting to land on the same spot each time. In fact, in each of n successive jumps he always lands on a fixed straight line but when he lands from the ith jump ($i = 1, 2, \ldots, n$) his displacement from the point from which he jumped is X_i cm, where X_i has the distribution $(*)$. His displacement from his starting point after n jumps is Y cm (so that $Y = \sum_{i=1}^{n} X_i$). Each jump is independent of the others.

Obtain the moment generating function for $Y/\sqrt{2n}$ and, by considering its logarithm, show that this moment generating function tends to $\exp(\frac{1}{2}\theta^2)$ as $n \to \infty$.

Given that $\exp(\frac{1}{2}\theta^2)$ is the moment generating function of the standard Normal random variable, estimate the least number of jumps such that there is a 5% chance that the frog lands 25 cm or more from his starting point.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself The density $\frac{1}{2}e^{-|x|}$ is symmetric. What does it look like? What is its mean?

STAGE 2 The frog's displacement

Ask yourself What is the moment generating function of $Y = X_1 + \cdots + X_n$? Of $Y/\sqrt{2n}$?

STAGE 3 The limit

Ask yourself Take the logarithm and use $\ln(1-u) \approx -u$ for small u.

STAGE 4 The frog's chances

Ask yourself For large n, Y is approximately $\mathrm{N}(0, 2n)$. When is $\mathrm{P}(|Y| \geq 25) = 0.05$?

Try this next

1. **Exact for two jumps.** Show that $Y = X_1 + X_2$ has density $\frac{1}{4}(1+|y|)e^{-|y|}$, and check its variance is 4.

2. **Binomial to normal.** For $X \sim \mathrm{B}(1, \frac{1}{2})$ shifted to have mean 0 (taking values $\pm\frac{1}{2}$), find $\mathrm{M}_X(\theta) = \cosh\frac{\theta}{2}$, and show that the moment generating function of a standardised sum of n such variables tends to $e^{\theta^2/2}$.

3. **The frog, exactly.** Use a computer (or the exact density for $n = 2$ as a check) to see how accurate the normal approximation is for $n = 82$.

4. **Why not Cauchy?** The Cauchy distribution (Section 12.22) has no moment generating function. Which step of the argument fails, and what happens to the average of n Cauchy variables?

Further reading

- **Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction*** (Oxford University Press), the chapter on generating functions and limit theorems.

- **Hans Fischer, *A History of the Central Limit Theorem*** (Springer). From Laplace to Lindeberg.
- **David Spiegelhalter, *The Art of Statistics*** (Pelican), on why the normal distribution is everywhere.

The full guided solution, and what the question leads to at university, is in Section 12.24:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/clt.html