

12.23 Stirling from Probability

WORKSHEET

The question

- (i) The random variable X has a binomial distribution with parameters n and p , where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, that

$$P(X = 8) \approx \frac{1}{2\sqrt{2\pi}}.$$

- (ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}}.$$

- (iii) By considering a Poisson distribution with parameter n , show that

$$n! \approx \sqrt{2\pi n} e^{-n} n^n.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** If a discrete random variable X with mean μ and variance σ^2 is approximately normal, roughly what is $P(X = \mu)$? (Think of the probability of an interval of width 1 centred at μ .)

STAGE 2 The central binomial coefficient

- ?** **Ask yourself** For $B(2n, \frac{1}{2})$, what are the mean and standard deviation? Write $P(X = n)$ exactly and approximately.

STAGE 3 Stirling's formula

- ?** **Ask yourself** A Poisson distribution with parameter n has mean and variance both equal to n . Write $P(X = n)$ exactly and approximately.

Try this next

- 1 Check the formula.** Use Stirling's formula to estimate $10!$ and $20!$, and compare with the true values 3 628 800 and 2 432 902 008 176 640 000.
- 2 From (iii) to (ii).** Show that Stirling's formula in part (iii) implies the approximation in part (ii).
- 3 Digits of $100!$.** Use $\log_{10} n! \approx n \log_{10} \frac{n}{e} + \frac{1}{2} \log_{10}(2\pi n)$ to show that $100!$ has 158 digits.
- 4 A tie in coin tossing.** Use part (ii) to show that the probability of exactly 500 heads in 1000 tosses of a fair coin is about 0.025.

Further reading

- **William Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1** (Wiley), Chapter II, which proves Stirling's formula.
- **Ronald Graham, Donald Knuth & Oren Patashnik, *Concrete Mathematics*** (Addison-Wesley), Chapter 9, on asymptotics.
- **Stephen Stigler, *The History of Statistics*** (Harvard University Press), on de Moivre and the birth of the normal curve.

The full guided solution, and what the question leads to at university, is in Section 12.23:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/stirling-probability.html