

12.22 A Distribution With No Mean

WORKSHEET

The question

The random variables X and Y are independently normally distributed with means 0 and variances 1. Show that the joint probability density function for (X, Y) is

$$f(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad -\infty < x < \infty, -\infty < y < \infty.$$

If (x, y) are the coordinates, referred to rectangular axes, of a point in the plane, explain what is meant by saying that this density is radially symmetrical.

The random variables U and V have a joint probability density function which is radially symmetrical (in the above sense). By considering the straight line with equation $U = kV$, or otherwise, show that

$$P\left(\frac{U}{V} < k\right) = 2P(U < kV, V > 0).$$

Hence, or otherwise, show that the probability density function of U/V is

$$g(k) = \frac{1}{\pi(1+k^2)} \quad -\infty < k < \infty.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** If (U, V) is a random point whose density depends only on its distance from the origin, what can you say about the angle Θ that the line from the origin to the point makes with an axis?

STAGE 2 Using the symmetry

- ?** **Ask yourself** $\frac{U}{V} < k$ happens in two ways, according to the sign of V . How are they related by the rotation through 180° , $(u, v) \mapsto (-u, -v)$?

STAGE 3 An angle

- ?** **Ask yourself** Draw the region $\{V > 0, U < kV\}$ in the plane with u horizontal and v vertical. It is a wedge with its vertex at the origin. What is its angle?

Try this next

- 1 The torch.** A torch at distance 1 from a long straight wall points in a direction making angle Θ with the perpendicular, where Θ is uniform on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Show that the spot's position $\tan \Theta$ has the Cauchy distribution.
- 2 The reciprocal.** Show that if C is Cauchy, so is $\frac{1}{C}$. (*Hint:* $\frac{U}{V}$ and $\frac{V}{U}$.)
- 3 The median works.** Show that the median of the Cauchy distribution is 0 and its quartiles are ± 1 .
- 4 Radius and angle.** For X, Y independent standard normals, show by changing to polar coordinates that $R^2 = X^2 + Y^2$ is exponential with mean 2 and is independent of the angle. (This is the basis of the Box–Muller method for generating normal random numbers.)

Further reading

- **Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction*** (Oxford University Press), on transformations of random variables.
- **Nassim Nicholas Taleb, *The Black Swan*** (Penguin). Heavy tails and why they matter, for a general reader.
- **Stephen Stigler, *The History of Statistics*** (Harvard University Press). Includes the

Cauchy–Bienaymé dispute.

The full guided solution, and what the question leads to at university, is in Section 12.22:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/cauchy.html