

12.21 Breaking a Stick

WORKSHEET

The question

- (i) A rod of unit length is cut into pieces of length X and $1 - X$; the latter is then cut in half. The random variable X is uniformly distributed over $[0, 1]$. For some values of X a triangle can be formed from the three pieces of the rod. Show that the conditional probability that, if a triangle can be formed, it will be obtuse-angled is $3 - 2\sqrt{2}$.
- (ii) The bivariate distribution of the random variables X and Y is uniform over the triangle with vertices $(1, 0)$, $(1, 1)$ and $(0, 1)$. A pair of values x, y is chosen at random from this distribution and a (perhaps degenerate) triangle ABC is constructed with $BC = x$ and $CA = y$ and $AB = 2 - x - y$. Show that the construction is always possible and that $\angle ABC$ is obtuse if and only if

$$y > \frac{x^2 - 2x + 2}{2 - x}.$$

Deduce that the probability that $\angle ABC$ is obtuse is $3 - 4 \ln 2$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** Three lengths form a triangle if and only if each is less than the sum of the other two. If the lengths add up to L , what does this say about the longest one? When is the angle opposite side c obtuse?

STAGE 2 The second construction

- ?** **Ask yourself** Check the three triangle inequalities for sides x , y and $2 - x - y$, for (x, y) in the triangle with vertices $(1, 0)$, $(1, 1)$, $(0, 1)$.

- ?** **Ask yourself** The angle at B is opposite the side $CA = y$. Write the condition $y^2 > x^2 + (2 - x - y)^2$ as an inequality for y .

STAGE 3 The area

- ?** **Ask yourself** Divide $x^2 - 2x + 2$ by $2 - x$. Then find the area of the part of the region above the curve, and divide by the region's area, $\frac{1}{2}$.

Try this next

- 1 The classic.** Show that if a stick is broken at two independent uniform points, the probability that the pieces form a triangle is $\frac{1}{4}$.
- 2 Break the longer piece.** Break a stick at a uniform point, then break the *longer* piece at a uniform point. Show that the probability that the three pieces form a triangle is $2 \ln 2 - 1 \approx 0.386$.
- 3 Acute or obtuse.** In part (ii), find the probability that the triangle ABC has an obtuse angle at C (opposite $AB = 2 - x - y$). Hence find the probability that it has an obtuse angle somewhere.
- 4 Expected longest piece.** For the classic broken stick, show that the expected length of the longest piece is $\frac{11}{18}$.

Further reading

- **Martin Gardner**, *The Colossal Book of Mathematics* (Norton), which includes the broken-stick problem.

- ▶ **Frederick Mosteller**, *Fifty Challenging Problems in Probability* (Dover).
- ▶ **Lewis Carroll**, *Pillow Problems* (Dover, with *A Tangled Tale*). Victorian puzzles, including random triangles.

The full guided solution, and what the question leads to at university, is in Section 12.21:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/broken-stick.html