

12.20 Buffon's Needle

WORKSHEET

The question

By making the substitution $y = \cos^{-1} t$, or otherwise, show that

$$\int_0^1 \cos^{-1} t \, dt = 1.$$

A pin of length $2a$ is thrown onto a floor ruled with parallel lines equally spaced at a distance $2b$ apart. The distance X of its centre from the nearest line is a uniformly distributed random variable taking values between 0 and b and the acute angle Y the pin makes with a direction perpendicular to the line is a uniformly distributed random variable taking values between 0 and $\pi/2$. X and Y are independent. If $X = x$ what is the probability that the pin crosses the line?

If $a < b$ show that the probability that the pin crosses a line for a general throw is $\frac{2a}{\pi b}$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself The pin's centre is at distance x from the nearest line, and the pin makes angle y with the perpendicular to the lines. How far does the pin reach towards the line?

STAGE 2 Given the position

Ask yourself For fixed x , which angles y give a crossing? What is their probability?

STAGE 3 Averaging over the position

Pause and try With a needle exactly as long as the half-spacing ($a = \frac{b}{2}$, so the needle's length $2a$ equals b), the crossing probability is $\frac{1}{\pi}$. How many throws would you need to estimate π to two decimal places?

Try this next

- A long needle.** If $a > b$, the needle may cross more than one line. Show that the expected number of crossings is still $\frac{2a}{\pi b}$.
- The noodle.** Bend a needle of length $2a$ into a right angle (two arms of length a). What is the expected number of crossings?
- Square tiles.** A coin of radius r is tossed onto a floor of square tiles of side $s > 2r$. Find the probability that it lies entirely within one tile.
- Lazzarini's needle.** With $a = \frac{5}{6}b$ (so that $\frac{2a}{\pi b} = \frac{5}{3\pi}$), show that 3408 throws with 1808 crossings give the estimate $\pi \approx \frac{355}{113}$.

Further reading

- Daniel Klain & Gian-Carlo Rota, *Introduction to Geometric Probability* (Cambridge University Press), which starts from Buffon's needle.
- Lee Badger, "Lazzarini's lucky approximation of π ", *Mathematics Magazine* 67 (1994). The statistics of a suspicious experiment.
- Paul Nahin, *Digital Dice* (Princeton University Press). Monte Carlo methods for probability problems.

The full guided solution, and what the question leads to at university, is in Section 12.20:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/buffon.html