

12.19 Random Points in the Plane

WORKSHEET

The question

In this question, you may assume that $\int_0^\infty e^{-x^2/2} dx = \sqrt{\frac{1}{2}\pi}$.

The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y .

The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $P(Y < y)$ and hence show that the probability density function of Y is $2\pi y k e^{-\pi k y^2}$ for $y \geq 0$.

Find $E(Y)$ and show that $\text{Var}(Y) = \frac{4 - \pi}{4\pi k}$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself A Poisson random variable with mean μ takes the value 0 with what probability?

STAGE 2 The distribution of the nearest distance



Ask yourself $Y \geq y$ means there is no supermarket within distance y . So what is $P(Y < y)$?

STAGE 3 The mean



Ask yourself Either integrate $y f(y)$ by parts, or use $E(Y) = \int_0^\infty P(Y > y) dy$. Then substitute to use the given Gaussian integral.

Try this next

- 1 The median.** Find the median distance to the nearest supermarket, and compare it with the mean.
- 2 The second nearest.** Find the distribution of the distance to the *second* nearest supermarket. (*Hint:* $P(Y_2 > y)$ is the probability of at most one supermarket within y .)
- 3 Normal coordinates.** If U and V are independent $N(0, \sigma^2)$, show that $\sqrt{U^2 + V^2}$ has density $\frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)}$. Which value of σ gives the supermarket distribution?
- 4 Clark–Evans.** In a survey of a 1 km^2 wood containing 400 trees, the mean distance from a tree to its nearest neighbour is 35 m. Is this consistent with complete spatial randomness, or does it suggest clustering or spacing?

Further reading

- **Peter Diggle, *Statistical Analysis of Spatial Point Patterns*** (Arnold). The standard reference, with the Clark–Evans test and its successors.
- **John Kingman, *Poisson Processes*** (Oxford University Press). A beautiful short book on Poisson processes in any number of dimensions.
- **Geoffrey Grimmett & David Stirzaker, *Probability and Random Processes*** (Oxford University Press), the chapter on the Poisson process.

The full guided solution, and what the question leads to at university, is in Section 12.19:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/spatial-poisson.html