

12.18 Memorylessness

WORKSHEET

The question

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant. The random variable Y is the greatest integer less than or equal to X , and $Z = X - Y$.

- (i) Show that, for any non-negative integer n , $P(Y = n) = (1 - e^{-\lambda})e^{-n\lambda}$.
- (ii) Show that $P(Z < z) = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}$ for $0 \leq z \leq 1$.
- (iii) Evaluate $E(Z)$.
- (iv) Obtain an expression for $P(Y = n \text{ and } z_1 < Z < z_2)$, where $0 \leq z_1 < z_2 \leq 1$ and n is a non-negative integer. Determine whether Y and Z are independent.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** What is $P(X > x)$ for $x \geq 0$? What is $P(X > s + t \mid X > s)$?

STAGE 2 The fractional part

? **Ask yourself** $Z < z$ means X lies in one of the intervals $[n, n + z)$. Add up their probabilities.

STAGE 3 The mean of Z

? **Ask yourself** For a random variable on $[0, 1]$, $E(Z) = \int_0^1 P(Z > z) dz$. Use it.

STAGE 4 Independence

? **Ask yourself** Compare with $P(Y = n) P(z_1 < Z < z_2)$, using parts (i) and (ii).

Try this next

- 1 Two bus routes.** (From STEP III, 2002.) Show that the minimum of independent exponentials with parameters λ_1 and λ_2 is exponential with parameter $\lambda_1 + \lambda_2$. Buses on route A and route B have exponential gaps with means 15 and 30 minutes, independently. Having just seen one of each leave, show that the median wait for the next bus is about 7 minutes.
- 2 Memoryless discrete.** Show that the geometric distribution is memoryless: $P(Y \geq m + n \mid Y \geq m) = P(Y \geq n)$.
- 3 The waiting-time paradox.** Buses arrive as a Poisson process with rate λ . Show that the gap containing a fixed time t (for large t) has expected length $\frac{2}{\lambda}$, twice the average gap.
- 4 The density of Z .** Find the probability density of Z , and sketch it for $\lambda = 1$ and $\lambda = 5$.

Further reading

- **Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction*** (Oxford University Press), on the exponential distribution and the Poisson process.

- **Sheldon Ross**, *Introduction to Probability Models* (Academic Press). Exponential distributions, Poisson processes and queues.
- **David Spiegelhalter**, *The Art of Uncertainty* (Pelican). Randomness and waiting in everyday life.

The full guided solution, and what the question leads to at university, is in Section 12.18:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/memoryless.html