

12.17 Banach's Matchboxes

WORKSHEET

The question

A taxi driver keeps a packet of toffees and a packet of mints in her taxi. From time to time she takes either a toffee (with probability p) or mint (with probability $q = 1 - p$). At the beginning of the week she has n toffees and m mints in the packets. On the N th occasion that she reaches for a sweet, she discovers (for the first time) that she has run out of that kind of sweet. What is the probability that she was reaching for a toffee?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself For the driver to discover on the N th reach that the toffees have run out, what must the N th reach be? What must the previous $N - 1$ reaches contain?

STAGE 1 Discovering the toffees are gone



Ask yourself In what order can the first $N - 1$ reaches occur? What condition on $N - 1 - n$ ensures she never found the mint packet empty?

STAGE 3 The conditional probability



Ask yourself We are told that a discovery happens on the N th reach. What is the probability that it was a toffee?



Pause and try Check the formula when $n = m$ and $p = q = \frac{1}{2}$. By symmetry the answer should be $\frac{1}{2}$ for every possible N . Is it?

Try this next

- 1 **Small case.** With $n = m = 1$ and $p = \frac{1}{2}$, list the possible sequences and find the distribution of N .
- 2 **Banach's formula.** Verify that $\sum_{k=0}^n \binom{2n-k}{n} 2^{-(2n-k)} = 1$ for $n = 1$ and $n = 2$.
- 3 **Equal tops.** Show that $P(K = 0) = P(K = 1)$ for every n in Banach's problem.
- 4 **Which runs out first?** For the taxi driver, find the probability that the toffees are the first packet to be discovered empty (at any time), when $n = m = 2$ and $p = \frac{2}{3}$.

Further reading

- William Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1 (Wiley), Chapter VI, which introduced Banach's problem to the world.
- R. Daniel Mauldin (ed.), *The Scottish Book* (Birkhäuser). The problems from the Scottish Café, with commentary.
- Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction* (Oxford University

Press), on the negative binomial distribution.

The full guided solution, and what the question leads to at university, is in Section 12.17:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/matchboxes.html