

12.16 Markov Chains Forwards and Backwards

WORKSHEET

The question

It has been observed that Professor Ecks proves three types of theorems: 1, those that are correct and new; 2, those that are correct, but already known; 3, those that are false. It has also been observed that, if a certain of her theorems is of type i , then her next theorem is of type j with probability p_{ij} , where p_{ij} is the entry in the i th row and j th column of the following array:

$$\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Let a_i , $i = 1, 2, 3$, be the probability that a given theorem is of type i , and let b_j be the consequent probability that the next theorem is of type j .

- (i) Explain why $b_j = a_1 p_{1j} + a_2 p_{2j} + a_3 p_{3j}$.
- (ii) Find values of a_1 , a_2 and a_3 such that $b_i = a_i$ for $i = 1, 2, 3$.
- (iii) For these values of the a_i find the probabilities q_{ij} that, if a particular theorem is of type j , then the *preceding* theorem was of type i .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What does the condition “each row adds up to 1” mean? Do the columns add up to 1?

STAGE 2 The stationary distribution



Ask yourself We want $\mathbf{a} = \mathbf{a}P$ with $a_1 + a_2 + a_3 = 1$. Write out the equations. Which are the simplest?

STAGE 3 Running the chain backwards



Ask yourself $q_{ij} = P(\text{previous } i \mid \text{current } j)$. Use Bayes’ theorem, with the stationary distribution for both the previous and the current theorem.

Try this next

- 1 **A rabbit warren.** (From STEP III, 2003.) Chambers A, B, C, D are at the corners of a square, joined by burrows along the sides, and each has a tunnel to the surface. A rabbit in a chamber runs to one of the two neighbouring chambers, or leaves by the tunnel, each with probability $\frac{1}{3}$. Let p_X be the probability of leaving by the tunnel at A , starting from chamber X . Show that $p_A = \frac{1}{3} + \frac{1}{3}p_B + \frac{1}{3}p_D$, and find p_A . Find the probability that a rabbit starting at A never visits C , given that it leaves by the tunnel at A .
- 2 **Two steps.** If Professor Ecks’s last theorem was new and correct, find the probabilities of the three types for the theorem after next.
- 3 **Eigenvalues.** Show that the eigenvalues of P are 1, 0.2 and 0.1.
- 4 **Reversible?** Check whether $a_i p_{ij} = a_j p_{ji}$ for Professor Ecks’s chain. What does the answer tell you?

Further reading

- **James Norris, *Markov Chains*** (Cambridge University Press). The standard undergraduate text.
- **Olle Häggström, *Finite Markov Chains and Algorithmic Applications*** (Cambridge University Press). Short, and good on Markov chain Monte Carlo.

- **Brian Hayes**, “**First links in the Markov chain**”, *American Scientist* 101 (2013). Markov’s Pushkin analysis, retold.

The full guided solution, and what the question leads to at university, is in Section 12.16:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/markov.html