

12.15 Extinction or Explosion

WORKSHEET

The question

The life of a certain species of elementary particles can be described as follows. Each particle has a life time of T seconds, after which it disintegrates into X particles of the same species, where X is a random variable with binomial distribution $B(2, p)$. A population of these particles starts with the creation of a single such particle at $t = 0$. Let X_n be the number of particles in existence in the time interval $nT < t < (n+1)T$, where $n = 1, 2, \dots$

Show that $P(X_1 = 2 \text{ and } X_2 = 2) = 6p^4q^2$, where $q = 1 - p$. Find the possible values of p if it is known that $P(X_1 = 2 \mid X_2 = 2) = 9/25$.

Explain briefly why $E(X_n) = 2p E(X_{n-1})$ and hence determine $E(X_n)$ in terms of p . Show that for one of the values of p found above $\lim_{n \rightarrow \infty} E(X_n) = 0$ and that for the other $\lim_{n \rightarrow \infty} E(X_n) = +\infty$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself If there are two particles at one stage, what is the distribution of the number at the next stage?

STAGE 2 Reasoning backwards



Ask yourself In how many ways can $X_2 = 2$ happen? (What can X_1 be?)

STAGE 3 The expected population



Ask yourself If there are k particles at stage $n - 1$, what is the expected number at stage n ? Then average over k .

Try this next

- 1 **Extinction in general.** For offspring distribution $B(2, p)$, show that the extinction probability is 1 if $p \leq \frac{1}{2}$ and $(\frac{q}{p})^2$ if $p > \frac{1}{2}$.
- 2 **The variance.** Show that $\text{Var}(X_n)$ satisfies $\text{Var}(X_n) = (2p)^2 \text{Var}(X_{n-1}) + 2pq \text{E}(X_{n-1})$, and find it for $p = \frac{1}{2}$.
- 3 **Extinction by generation n .** Let $e_n = P(X_n = 0)$. Show that $e_n = G(e_{n-1})$, and compute e_1, e_2, e_3 for $p = \frac{3}{4}$. Do they approach $\frac{1}{9}$?
- 4 **Surnames.** If each man has 0, 1, 2 or 3 sons with probabilities $\frac{1}{4}$ each, what is the probability that his surname eventually dies out?

Further reading

- **Geoffrey Grimmett & David Stirzaker, *Probability and Random Processes*** (Oxford University Press), the section on branching processes.
- **Adam Kucharski, *The Rules of Contagion*** (Profile). Reproduction numbers and outbreaks, for a general reader.
- **David Kendall, “Branching processes since 1873”, *Journal of the London Mathematical Society* 41 (1966).** The history, including Bienaymé.

The full guided solution, and what the question leads to at university, is in Section 12.15:

https://keithmcnulty.github.io/step_book_mcnulty/ch12/branching.html