12.14 Mixed Strategies

WORKSHEET

The question

A and B play a guessing game. Each simultaneously names one of the numbers $1, 2, 3$. If the numbers differ by 2, whoever guessed the *smaller* pays the opponent £2. If the numbers differ by 1, whoever guessed the *larger* pays the opponent £1. Otherwise no money changes hands. Many rounds of the game are played.

(i) If A says he will always guess the same number N, explain (for each value of N) how B can maximise his winnings.

(ii) In an attempt to improve his play, A announces that he will guess each number at random with probability $\frac{1}{3}$, guesses on different rounds being independent. To counter this, B secretly decides to guess j with probability b_j ($j = 1, 2, 3$, $b_1 + b_2 + b_3 = 1$), guesses on different rounds being independent. Derive an expression for B's expected winnings on any round. How should the probabilities b_j be chosen so as to maximize this expression?

(iii) A now announces that he will guess j with probability a_j ($j = 1, 2, 3$, $a_1 + a_2 + a_3 = 1$). If B guesses j with probability b_j ($j = 1, 2, 3$, $b_1 + b_2 + b_3 = 1$), obtain an expression for his expected winnings in the form

$$Xa_1 + Ya_2 + Za_3.$$

Show that he can choose b_1, b_2 and b_3 such that X, Y and Z are all non-negative. Deduce that, whatever values for a_j are chosen by A, B can ensure that in the long run he loses no money.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Write down a table of B's winnings for each pair of guesses (B's guess down the side, A's across the top).

STAGE 2 Against random guessing

Ask yourself If A guesses uniformly at random, what is B's expected gain when B says j? (Average along row j of the table.)

STAGE 3 The general case

Ask yourself Now A plays j with probability a_j. Collect B's expected winnings by A's guess: what is the coefficient of a_1 (the average of column 1 with weights b_j)?

Ask yourself Can you make all three zero? (Three equations, but they are not independent; why not?)

Try this next

1. **A's best reply.** If B uses $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, show that every strategy of A gives expected winnings 0. If instead B uses $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, find A's best reply.
2. **Rock, paper, scissors.** Write the payoff matrix for rock–paper–scissors and find the optimal strategy. Why is it uniform here but not in the question?
3. **Uniqueness.** Show that $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ is the *only* strategy for B that guarantees not losing. (*Hint:* it must make X, Y, Z all non-negative; use $X + 2Y + Z = 0$.)
4. **A modified game.** If the penalty for differing by 2 is £3 instead of £2, find the optimal strategy.

Further reading

- **Ken Binmore, *Playing for Real*** (Oxford University Press). A thorough but friendly introduction to game theory.
- **Avinash Dixit & Barry Nalebuff, *The Art of Strategy*** (Norton). Game theory with examples from sport and business.

- **William Poundstone, *Prisoner's Dilemma*** (Anchor). Von Neumann, game theory and the Cold War.

The full guided solution, and what the question leads to at university, is in Section 12.14:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/mixed-strategies.html