

# 12.12 When to Stop Looking

## WORKSHEET

### The question

On the basis of an interview, the  $N$  candidates for admission to a college are ranked in order according to their mathematical potential. The candidates are interviewed in random order (that is, each possible order is equally likely).

- (i) Find the probability that the best amongst the first  $n$  candidates interviewed is the best overall.
- (ii) Find the probability that the best amongst the first  $n$  candidates interviewed is the best or second best overall.

Verify your answers for the case  $N = 4$ ,  $n = 2$  by listing the possibilities.

## Hints

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Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



**Ask yourself** In a random order, what is the probability that the overall best candidate is among the first  $n$  interviewed?

### STAGE 2 Best or second best



**Ask yourself** The best of the first  $n$  is the second best overall exactly when the second best is among the first  $n$  but the best is not. What is that probability?

### STAGE 3 The optimal strategy



**Ask yourself** Strategy: reject the first  $r$  candidates, then accept the first candidate better than all of them. If the best candidate is in position  $k > r$ , when does the strategy pick them?



**Ask yourself** For large  $N$ , write  $x = \frac{r}{N}$ . The sum is approximately  $\ln \frac{N}{r}$ . What value of  $x$  maximises  $x \ln \frac{1}{x}$ ?

## Try this next

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- 1 **Ice-creams.** (From STEP II, 2006.) Ice-creams of  $n$  different sizes are offered in random order, and you must choose one. You reject the first and take the first one bigger than it (or the last, if the first was the biggest). Show that the probability of getting the biggest is  $\frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n-1})$ , which is  $\frac{11}{24}$  for  $n = 4$ . For  $n = 4$ , find the probabilities of getting the second, third and fourth biggest.
- 2 **Small  $N$ .** For  $N = 4$ , find the best value of  $r$  and the probability of success. Check by listing.
- 3 **Best or second best.** Using part (ii), argue that “the best of the first  $n$  is one of the top two” is much more likely than “the best of the first  $n$  is the best”. What does this suggest about strategies for getting one of the top two?
- 4 **An exact maximum.** For  $N = 10$ , check from the figure that  $r = 3$  is optimal. For  $N = 100$ , show that  $r = 37$  is optimal (to four decimal places the success probabilities for  $r = 36, 37, 38$  are 0.3710, 0.3710, 0.3708).

## Further reading

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- ▶ **Thomas Ferguson**, “Who solved the secretary problem?”, *Statistical Science* 4 (1989). History and variations; free online.
- ▶ **Brian Christian & Tom Griffiths**, *Algorithms to Live By* (William Collins). Optimal stopping and other algorithms applied to everyday decisions.
- ▶ **Martin Gardner**, *New Mathematical Diversions* (Mathematical Association of America). The game of googol.

The full guided solution, and what the question leads to at university, is in Section 12.12:  
[https://keithmcnulty.github.io/step\\_book\\_mcnulty/ch12/secretary.html](https://keithmcnulty.github.io/step_book_mcnulty/ch12/secretary.html)