

12.11 Stamps, Cups and e

WORKSHEET

The question

I have a Penny Black stamp which I want to sell to my friend Jim, but we cannot agree a price. So I put the stamp under one of two cups, jumble them up, and let Jim guess which one it is under. If he guesses correctly, I add a third cup, jumble them up, and let Jim guess correctly, adding another cup each time. The price he pays for the stamp is $\pounds N$, where N is the number of cups present when Jim fails to guess correctly. Find $P(N = k)$. Show that $E(N) = e$ and calculate $\text{Var}(N)$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? Ask yourself For $N = 4$, what must happen? With what probability?

STAGE 2 The expectation

? Ask yourself Simplify $k \cdot \frac{k-1}{k!}$. What series do you get?

STAGE 3 The variance

? Ask yourself Find $E(N^2)$ by writing $k^2 = k(k-1) + k$, or by simplifying $\frac{k^2(k-1)}{k!}$ directly.

Try this next

- 1 The generating function.** Show that $E(t^N) = 1 + (t-1)e^t$, and use it to find $E(N)$ and $E(N(N-1))$.
- 2 The tail formula.** Prove that $E(N) = \sum_{k \geq 0} P(N > k)$ for a random variable taking non-negative integer values, and use it to find $E(N)$ in one line.
- 3 Starting with one cup.** If the game starts with one cup (which Jim cannot get wrong), how does the distribution change? What is the expected price?
- 4 The simplex.** Show that the volume of $\{u_1 + u_2 + u_3 < 1, u_i \geq 0\}$ is $\frac{1}{6}$, and use induction to show the k -dimensional version is $\frac{1}{k!}$.

Further reading

- **Eli Maor, *e: The Story of a Number*** (Princeton University Press). The history of e , including its appearances in probability.
- **Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction*** (Oxford University Press), on generating functions.
- **Paul Nahin, *Digital Dice*** (Princeton University Press). Probability puzzles solved by Monte Carlo simulation.

The full guided solution, and what the question leads to at university, is in Section 12.11:

https://keithmcnulty.github.io/step_book_mcnulty/ch12/stamps-and-e.html