

12.10 Tying Loose Ends

WORKSHEET

The question

A box contains n pieces of string, each of which has two ends. I select two string ends at random and tie them together. This creates either a ring (if the two ends are from the same string) or a longer piece of string. I repeat the process of tying together string ends chosen at random until there are none left.

Find the expected number of rings created at the first step and hence obtain an expression for the expected number of rings created by the end of the process. Find also an expression for the variance of the number of rings created.

Given that $\ln 20 \approx 3$ and that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n$ for large n , determine approximately the expected number of rings created in the case $n = 40\,000$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** After tying two ends together, how many pieces of string are left (counting a ring as no longer a “piece”)? Does it depend on whether a ring was made?

STAGE 2 All the knots

? **Ask yourself** By the observation above, what is the situation before the second knot? The third? Let I_k indicate a ring at the knot made when k pieces remain. What are the $E(I_k)$?

STAGE 3 The variance

? **Ask yourself** Are the indicators I_k independent? (Does the probability of a ring at one step depend on what happened earlier?)

STAGE 4 Forty thousand strings

? **Ask yourself** Write the sum of the reciprocals of the odd numbers up to $2n - 1$ using harmonic numbers.

Try this next

- 1 Small cases.** For $n = 2$, list the possibilities and check that the expected number of rings is $\frac{4}{3}$.
- 2 One big loop.** Show that the probability that the process ends with a single loop containing all the strings is $\prod_{k=2}^n (1 - \frac{1}{2k-1})$, and that this is approximately $\sqrt{\frac{\pi}{4n}}$ for large n . (*Hint*: relate the product to $\binom{2n}{n}$.)
- 3 Cycles of permutations.** Show that the expected number of cycles of a random permutation of n objects is H_n , by building the permutation one element at a time (the “Chinese restaurant” construction).
- 4 The variance.** Show that $\text{Var}(R) = E(R) - \sum_{k=1}^n \frac{1}{(2k-1)^2}$, and that the subtracted sum tends to $\frac{\pi^2}{8}$.

Further reading

- ▶ **Peter Winkler, *Mathematical Puzzles: A Connoisseur's Collection*** (A K Peters), which includes the string-tying puzzle.
- ▶ **Richard Arratia, Andrew Barbour & Simon Tavaré, *Logarithmic Combinatorial Structures*** (European Mathematical Society). Cycles, primes and random structures, for later.
- ▶ **Colin Adams, *The Knot Book*** (American Mathematical Society). An introduction to knot theory, including random knots.

The full guided solution, and what the question leads to at university, is in Section 12.10:
https://keithmcnulty.github.io/step_book_mcnulty/ch12/loose-ends.html