

11.9 The Safety Parabola

WORKSHEET

The question

A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x - y plane, where the x -axis is horizontal, the y -axis is vertical and the origin is O . Obtain the Cartesian equation of the particle's trajectory in terms of u , g and λ , where $\lambda = \tan \theta$.

Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x , the coordinate y can take all values up to a maximum value, Y , which you should determine as a function of x , u and g .

Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u .

Hence find the furthest distance from O that can be achieved by such a projectile.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Write x and y in terms of t , and eliminate t . How can you write $\sec^2 \theta$ in terms of $\lambda = \tan \theta$?

STAGE 1 Fix x , vary the angle



Ask yourself For a fixed $x > 0$, think of y as a function of λ . What kind of function is it? Which values does it take as λ ranges over all real numbers?

STAGE 2 The region



Ask yourself Through a point strictly below the curve, how many trajectories pass? And through a point on it?

STAGE 3 The furthest point



Ask yourself The furthest reachable point from O (on or above the ground) lies on the boundary. Write the squared distance $x^2 + Y^2$ as a function of x^2 . Is it increasing or decreasing?

Try this next

- 1 An exploding firework.** (From STEP II, 2023.) An explosion at the origin sends particles in all directions with speed u . Show that at time t they lie on a sphere of radius ut whose centre falls freely from the origin. Show that a particle projected horizontally with speed u from the top of the paraboloid of safety moves along its surface for ever, and find when it must be launched to collide with the particle fired at angle α .
- 2 Firing up a slope.** Find the maximum range of a projectile with speed u up a plane inclined at angle β . (*Hint:* where does the line $y = x \tan \beta$ meet the parabola of safety?) Show that the best angle bisects the angle between the slope and the vertical.
- 3 Two angles.** Show that the two angles that reach a point P are symmetric about the bisector of the angle between OP and the vertical.
- 4 The foci.** Show that the trajectory with angle θ has its focus at $(\frac{u^2}{2g} \sin 2\theta, -\frac{u^2}{2g} \cos 2\theta)$, and hence that the foci lie on a circle about O .

Further reading

- ▶ **David Morin**, *Introduction to Classical Mechanics* (Cambridge University Press), Chapter 3, which discusses the envelope of trajectories.
- ▶ **J. W. Bruce, P. J. Giblin & C. G. Gibson**, “On caustics of plane curves”, *American Mathematical Monthly* 88 (1981). Envelopes and caustics in geometry.
- ▶ **E. H. Lockwood**, *A Book of Curves* (Cambridge University Press). Envelopes, caustics and many classical curves, with constructions.

The full guided solution, and what the question leads to at university, is in Section 11.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/safety-parabola.html