

11.8 Tilted Equilibria

WORKSHEET

The question

In this question, $n \geq 2$.

- (i) A solid, of uniform density, is formed by rotating through 360° about the y -axis the region bounded by the part of the curve $r^{n-1}y = r^n - x^n$ with $0 \leq x \leq r$, and the x - and y -axes. Show that the y -coordinate of the centre of mass of this solid is $\frac{nr}{2(n+1)}$.
- (ii) Show that the normal to the curve $r^{n-1}y = r^n - x^n$ at the point $(rp, r(1 - p^n))$, where $0 < p \leq 1$, meets the y -axis at $(0, Y)$, where $Y = r\left(1 - p^n - \frac{1}{np^{n-2}}\right)$.

In the case $n = 4$, show that the greatest value of Y is $\frac{1}{4}r$.

- (iii) A solid is formed by rotating through 360° about the y -axis the region bounded by the curves $r^3y = r^4 - x^4$ and $ry = -(r^2 - x^2)$, both for $0 \leq x \leq r$. A and B are the points $(0, -r)$ and $(0, r)$, respectively, on the surface of the solid.

Show that the solid can rest in equilibrium on a horizontal surface with the vector $\overrightarrow{AB}$ at three different, non-zero, angles to the upward vertical. You should not attempt to find these angles.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** A solid rests on a table, touching at a single point X of its curved surface. The forces on it are its weight (acting at the centre of mass G) and the reaction at X . When are these in equilibrium?

STAGE 1 The centre of mass

- ?** **Ask yourself** Slice the solid into thin cylindrical shells of radius x , thickness δx and height y . What is the volume of a shell, and where is its centre of mass?

STAGE 2 Where normals meet the axis

- ?** **Ask yourself** For $n = 4$, $Y = r(1 - p^4 - \frac{1}{4p^2})$. Where is it greatest? What happens as $p \rightarrow 0$, and at $p = 1$?

STAGE 3 The spinning top

- ?** **Ask yourself** The solid in part (iii) is a dome ($n = 4$) on top of an upside-down paraboloid ($n = 2$). Where is its centre of mass?

STAGE 4 Counting equilibria

- ?** **Ask yourself** A tilted equilibrium needs a point (not A or B) whose normal passes through G . On each part of the surface, how many points have normals meeting the axis at height $\frac{3r}{35}$?

Try this next

- 1 A self-righting toy.** (From STEP II, 1996.) A toy is made of a solid cone of height λa fixed to a solid hemisphere of radius a along their common circular face. Show that if $\lambda < \sqrt{3}$ the toy returns upright when placed with its hemisphere on a table, but that if $\lambda > \sqrt{3}$ it may overbalance. (*Hint:* where must G be relative to the centre of the hemisphere?) Show that if it is placed with the cone touching the table, it stays there whatever λ .
- 2 A sphere.** For a uniform solid sphere, every normal passes through the centre. What does that say about its equilibria? What if the sphere's centre of mass is displaced slightly from its centre?

- 3 An ellipse.** Show that an elliptical disc rolling in its own plane on a line has exactly four equilibria, and identify which are stable.
- 4 Two paraboloids.** Replace the dome in part (iii) by a paraboloid $ry = r^2 - x^2$, so that the solid is symmetric about the x -axis. Show that G is at the origin, find the tilted equilibria, and explain why they come in pairs with equal tilts.

Further reading

- ▶ **Péter Várkonyi & Gábor Domokos**, “Static equilibria of rigid bodies: dice, pebbles, and the Poincaré–Hopf theorem”, *Journal of Nonlinear Science* 16 (2006). The Gömböc paper.
- ▶ **Daniel Kleppner & Robert Kolenkow**, *An Introduction to Mechanics* (Cambridge University Press), on equilibrium and stability.
- ▶ **Chris Rorres**, “Completing Book II of Archimedes’s *On Floating Bodies*”, *Mathematical Intelligencer* 26 (2004). Archimedes’ stability analysis, with modern pictures.

The full guided solution, and what the question leads to at university, is in Section 11.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/tilted-equilibria.html