11.7 Simpson's Rule in a Rod

WORKSHEET

The question

A body of mass m and centre of mass O is said to be *dynamically equivalent* to a system of particles of total mass m and centre of mass O if the moment of inertia of the system of particles is the same as the moment of inertia of the body, about any axis through O. Show that this implies that the moment of inertia of the system of particles is the same as that of the body about *any* axis.

Show that a uniform rod of length $2a$ and mass m is dynamically equivalent to a suitable system of three particles, one at each end of the rod, and one at the midpoint.

Use this result to deduce that a uniform rectangular lamina of mass M is dynamically equivalent to a system consisting of particles each of mass $\frac{1}{36}M$ at the corners, particles each of mass $\frac{1}{9}M$ at the midpoint of each side, and a particle of mass $\frac{4}{9}M$ at the centre. Hence find the moment of inertia of a square lamina, of side $2a$ and mass M, about one of its diagonals.

The mass per unit length of a thin rod of mass m is proportional to the distance from one end of the rod, and a dynamically equivalent system consists of one particle at each end of the rod and one at the midpoint. Write down a set of equations which determines these masses, and show that, in fact, only two particles are required.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Two systems have the same mass, the same centre of mass, and the same moment of inertia about every axis through it. Why does the parallel axis theorem then give the same moment of inertia about every other axis too?

STAGE 1 The rod

Ask yourself Put masses p at each end and q at the middle. By symmetry the centre of mass is right. What do the total mass and the moment of inertia about a perpendicular axis through the centre require?

Ask yourself Does this work for *every* axis through the centre, not just the perpendicular one?

STAGE 2 The rectangle

Ask yourself Cut the lamina into thin strips parallel to one side. Replace each strip by its three equivalent particles. What do you get, and what can you do next?

STAGE 4 A non-uniform rod

Ask yourself Let the rod have length L and density $\frac{2mx}{L^2}$ at distance x from one end. Put masses p_0, p_1, p_2 at $x = 0$, $\frac{L}{2}$, L. Matching the total mass, the centre of mass and the moment of inertia about the end gives three equations. What are they?

Try this next

1 **Simpson's rule.** Use Simpson's rule with a single interval to estimate $\int_0^1 \frac{\mathrm{d}x}{1+x^2}$, and compare with $\frac{\pi}{4}$. Check that the rule is exact for $\int_{-1}^1 x^3\,\mathrm{d}x$ and $\int_{-1}^1 x^2\,\mathrm{d}x$ but not for $\int_{-1}^1 x^4\,\mathrm{d}x$.

2 **All axes alike.** Show that the moment of inertia of a square lamina about any axis through its centre in its plane is $\frac{1}{3}Ma^2$. (*Hint:* the nine equivalent particles are symmetric under rotation by 90°; what does that say about a quadratic expression

in $\cos\psi$ and $\sin\psi$?)

3 **A triangle.** Show that a uniform triangular lamina of mass M is dynamically equivalent to three particles of mass $\frac{1}{3}M$ at the midpoints of its sides.

4 **A solid cube.** Find a dynamically equivalent system of 27 particles for a uniform solid cube, and use it to find the moment of inertia of a cube of side $2a$ about an edge.

Further reading

- **Daniel Kleppner & Robert Kolenkow, *An Introduction to Mechanics*** (Cambridge University Press), the chapter on rigid bodies and moments of inertia.
- **Endre Süli & David Mayers, *An Introduction to Numerical Analysis*** (Cambridge University Press), the chapters on numerical integration, including Simpson's rule and Gaussian quadrature.
- **Philip Davis & Philip Rabinowitz, *Methods of Numerical Integration*** (Dover). The classic reference.

The full guided solution, and what the question leads to at university, is in Section 11.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/simpsons-rod.html