

11.5 Gravity in an Accelerating Train

WORKSHEET

The question

A train moves westwards on a straight horizontal track with constant acceleration a , where $a > 0$. Axes are chosen as follows: the origin is fixed in the train; the x -axis is in the direction of the track with the positive x -axis pointing to the East; and the positive y -axis points vertically upwards.

A smooth wire is fixed in the train. It lies in the x - y plane and is bent in the shape given by $ky = x^2$, where k is a positive constant. A small bead is threaded onto the wire. Initially, the bead is held at the origin. It is then released.

- (i) Explain why the bead cannot remain stationary relative to the train at the origin.
- (ii) Show that, in the subsequent motion, the coordinates (x, y) of the bead satisfy

$$\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$$

and deduce that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant during the motion.

- (iii) Find an expression for the maximum vertical displacement, b , of the bead from its initial position in terms of a , k and g .
- (iv) Find the value of x for which the speed of the bead relative to the train is greatest and give this maximum speed in terms of a , k and g .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** The train accelerates westwards (in the negative x direction) at a .
 ? For an observer in the train, what “fictitious” force appears to act on a mass m ?

STAGE 2 An energy equation

- Ask yourself** Write Newton’s law in the train’s frame, including the normal reaction N . Then take the dot product with the velocity $(\dot{x}, \dot{y})$. Why does N disappear?

STAGE 3 How high?

- Ask yourself** The constant is 0, since the bead starts at rest at the origin. At the highest point the bead is momentarily at rest. What equation does that give?

STAGE 4 The fastest point

- Ask yourself** Write the speed squared as a function of x alone. Where is it greatest?

- Ask yourself** Find the gradient of the wire at $x = \frac{ak}{2g}$. How does it relate to the direction of the effective gravity $(a, -g)$?

Try this next

- 1 A cannon on a train.** (From STEP I, 2000.) A railway carriage moves north with acceleration a , and a child fires a toy cannon southwards at angle θ and speed V . Show that the range on the carriage floor is greatest when $\tan 2\theta = -\frac{g}{a}$, that this angle is approximately $\frac{\pi}{4} + \frac{a}{2g}$ when a is small, and that the maximum range is then approximately $\frac{V^2}{g} + \frac{V^2 a}{g^2}$.
- 2 A headwind.** (From STEP II, 2006.) A projectile is fired north at speed u and angle θ , against a wind that exerts a constant southward force f per unit mass. Find where it lands. Show that if θ exceeds a certain angle α the wind blows it back towards O before it lands, and find α . Describe the motion when $f = g$ and $\theta = 45^\circ$.

- 3 Coffee in a train.** Show that the surface of a cup of coffee in the accelerating train is a plane tilted at angle $\arctan(a/g)$ to the horizontal.
- 4 Small oscillations.** Find the period of small oscillations of the bead about $x = \frac{ak}{2g}$. (*Hint:* near the new bottom the wire is approximately a circle of radius equal to its radius of curvature.)

Further reading

- **Daniel Kleppner & Robert Kolenkow, *An Introduction to Mechanics*** (Cambridge University Press), the chapter on non-inertial frames and fictitious forces.
- **Albert Einstein, *Relativity: The Special and the General Theory*** (various editions). Einstein's own popular account, including the accelerating box.
- **Pedro Ferreira, *The Perfect Theory*** (Little, Brown). A biography of general relativity.

The full guided solution, and what the question leads to at university, is in Section 11.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/accelerating-frame.html