

11.4 The Tautochrone

WORKSHEET

The question

A smooth, axially symmetric bowl has its vertical cross-sections determined by $s = 2\sqrt{ky}$, where s is the arc-length measured from its lowest point V , and y is the height above V . A particle is released from rest at a point on the surface at a height h above V . Explain why

$$\left(\frac{ds}{dt}\right)^2 + 2gy$$

is constant.

Show that the time for the particle to reach V is

$$\pi\sqrt{\frac{k}{2g}}.$$

Two elastic particles of mass m and αm , where $\alpha < 1$, are released simultaneously from opposite sides of the bowl at heights $\alpha^2 h$ and h respectively. If the coefficient of restitution between the particles is α , describe the subsequent motion.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself The bowl is smooth. Which forces act on the particle, and which of them does work?

STAGE 2 Simple harmonic motion in s



Ask yourself Use $s = 2\sqrt{ky}$ to replace y by a function of s . What kind of motion does the resulting equation describe?

STAGE 3 The time to the bottom



Pause and try Why does this work? The “restoring force” along the curve is g times the slope, $g \frac{dy}{ds} = \frac{g}{2k}s$: exactly proportional to the distance from the bottom, measured along the curve. That is the defining property of simple harmonic motion. A particle starting higher has further to go, but is pushed proportionally harder.

STAGE 4 Two particles



Ask yourself When do the particles reach V , and how fast are they moving? Who has more momentum?



Ask yourself With total momentum zero and coefficient of restitution α , what are the velocities just after the collision?

Try this next

- 1 Different heights.** A particle is released from height h in the tautochrone bowl. Show that its speed at V is $\sqrt{2gh}$, as it must be by energy conservation, and check that this agrees with ωs_0 .
- 2 An ordinary bowl.** In a hemispherical bowl of radius R , compare the time taken to reach the bottom from a small height with the time from the rim. Which is longer? (Use the result of Section 11.3.)
- 3 Equal masses.** Repeat the second part of the question with two particles of equal mass m , released from heights h_1 and h_2 , and coefficient of restitution e . Do they still always collide at V ?

- 4 **Cycloid arc length.** Show that one arch of a cycloid has length $8R$, and that the area under it is $3\pi R^2$, three times the area of the rolling circle.

Further reading

- ▶ **George Simmons**, *Differential Equations with Applications and Historical Notes* (CRC Press). Includes the tautochrone and the brachistochrone, with their history.
- ▶ **E. A. Whitman**, “**Some historical notes on the cycloid**”, *American Mathematical Monthly* 50 (1943). Why the cycloid was called “the Helen of geometers”.
- ▶ **Paul Nahin**, *When Least Is Best* (Princeton University Press). The brachistochrone and other optimisation problems, told with history.

The full guided solution, and what the question leads to at university, is in Section 11.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/tautochrone.html