

11.3 The Real Pendulum

WORKSHEET

The question

Consider a simple pendulum of length l and angular displacement θ , which is **not** assumed to be small. Show that

$$\frac{1}{2}l\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - \cos\gamma),$$

where γ is the maximum value of θ . Show also that the period P is given by

$$P = 2\sqrt{\frac{l}{g}} \int_0^\gamma (\sin^2(\gamma/2) - \sin^2(\theta/2))^{-\frac{1}{2}} d\theta.$$

By using the substitution $\sin(\theta/2) = \sin(\gamma/2) \sin\phi$, and then finding an approximate expression for the integrand using the binomial expansion, show that for small values of γ the period is approximately

$$2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16}\right).$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself The bob moves on a circle of radius l . What are its speed and its height (relative to the lowest point) in terms of θ and $\dot{\theta}$?

STAGE 2 The period as an integral



Ask yourself How can you write $\cos \theta - \cos \gamma$ in terms of half-angles? And how much of the period is spent going from $\theta = 0$ to $\theta = \gamma$?

STAGE 3 Removing the singularity



Ask yourself The integrand becomes infinite at $\theta = \gamma$. Write $k = \sin \frac{\gamma}{2}$ and substitute $\sin \frac{\theta}{2} = k \sin \phi$. What range of ϕ corresponds to $0 \leq \theta \leq \gamma$? What happens to the integrand?

STAGE 4 The first correction



Ask yourself For small k , expand $(1 - k^2 \sin^2 \phi)^{-1/2}$ by the binomial theorem. What is $\int_0^{\pi/2} \sin^2 \phi \, d\phi$?

Try this next

- 1 A tethered particle.** (From STEP III, 2015.) A particle of mass m on a smooth horizontal rail is attached to a peg at distance a from the rail by an elastic string of natural length a and modulus λ . Starting with the string just taut, it is given speed v along the rail. Show that $\dot{x}^2 = v^2 - k^2(\sqrt{x^2 + a^2} - a)^2$, where $k^2 = \frac{\lambda}{ma}$, find the greatest displacement, and show that for $v \ll ka$ the period is approximately $\sqrt{\frac{32a}{kv}} \int_0^1 \frac{du}{\sqrt{1-u^4}}$. Why does the period now *depend* on v , even for small oscillations?
- 2 The next term.** Carry the binomial expansion one term further and show that $P \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16} + \frac{11\gamma^4}{3072}\right)$.
- 3 The AGM by hand.** Use three steps of the AGM to estimate the period of a pendulum with amplitude 90° , and compare with the value 1.1803 in the figure.
- 4 Clocks.** A pendulum clock is regulated to keep perfect time with an amplitude of 4° . If its amplitude increases to 5° , how many seconds a day does it lose?

Further reading

- ▶ **David Morin**, *Introduction to Classical Mechanics* (Cambridge University Press), Chapter 4, which treats the large-amplitude pendulum.
- ▶ **Jonathan Borwein & Peter Borwein**, *Pi and the AGM* (Wiley). Gauss's discovery and its uses, including the fastest ways to compute π .
- ▶ **Dava Sobel**, *Longitude* (Fourth Estate). The story of timekeeping at sea, beyond the pendulum.

The full guided solution, and what the question leads to at university, is in Section 11.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/real-pendulum.html