

11.2 Normal Modes

WORKSHEET

The question

Two particles X and Y , of equal mass m , lie on a smooth horizontal table and are connected by a light elastic spring of natural length a and modulus of elasticity λ . Two more springs, identical to the first, connect X to a point P on the table and Y to a point Q on the table. The distance between P and Q is $3a$.

Initially, the particles are held so that $XP = a$, $YQ = \frac{1}{2}a$, and $PXYQ$ is a straight line. The particles are then released.

At time t , the particle X is a distance $a + x$ from P and the particle Y is a distance $a + y$ from Q . Show that

$$m \frac{d^2x}{dt^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving $\frac{d^2y}{dt^2}$. Deduce that

$$x - y = A \cos \omega t + B \sin \omega t$$

where A and B are constants to be determined and $ma\omega^2 = \lambda$. Find a similar expression for $x + y$.

Show that Y will never return to its initial position.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** With X at distance $a + x$ from P and Y at distance $a + y$ from Q , how long is the middle spring XY ? Which springs are stretched and which compressed when x and y are positive?

STAGE 2 Decoupling

- Ask yourself** Each equation involves both x and y . Which combinations of the two equations involve only one combination of x and y ?

STAGE 4 Never returning

- Ask yourself** Write y in terms of t . For Y to be back at its initial position, $y = -\frac{1}{2}a$. What must both cosines be?

Try this next

- Energy.** Show that the total energy $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2a}(x^2 + y^2 + (x + y)^2)$ is constant, and that it splits into a sum of two terms, one for each normal mode, each of which is separately constant.
- Pure modes.** How should the particles be started so that the motion is periodic, with period $\frac{2\pi}{\omega}$? With period $\frac{2\pi}{\sqrt{3}\omega}$?
- A stiffer middle spring.** If the middle spring has modulus μ instead of λ , find the two normal-mode frequencies. For which values of μ/λ is every motion periodic?
- Three masses.** Three equal masses are connected in a line by four identical springs between two fixed points. Find the matrix and its eigenvalues $(2 - \sqrt{2}, 2, 2 + \sqrt{2})$, and describe the three normal modes.

Further reading

- ▶ **A. P. French, *Vibrations and Waves*** (MIT Introductory Physics Series), Chapter 5, on coupled oscillators and normal modes.
- ▶ **Gilbert Strang, *Introduction to Linear Algebra*** (Wellesley-Cambridge), the chapter on eigenvalues, which includes springs and masses as a main example.
- ▶ **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press), Chapter 8, on quasi-

periodic motion on a torus.

The full guided solution, and what the question leads to at university, is in Section 11.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/normal-modes.html